Gorenstein Fano polytopes arising from two poset polytopes

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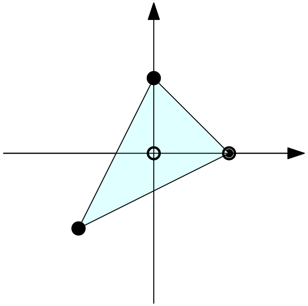
This talk is joint work with Takayuki Hibi.

Gorenstein Fano polytope

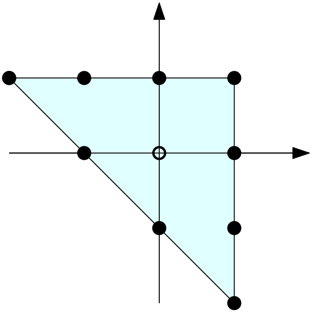
Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d.

- $\mathcal{P}$: Fano $\overset{\text{def}}{\Longleftrightarrow}$ the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$.
- $\mathcal{P}$: Gorenstein Fano (reflexive)
  $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{P}$ is Fano and its dual polytope
  $$\mathcal{P}^\vee := \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{y} \in \mathcal{P}\}$$
  is integral as well.
- $\mathcal{P}$: normal
  $\overset{\text{def}}{\Longleftrightarrow}$ for each integer $N > 0$ and for each $\mathbf{a} \in N\mathcal{P} \cap \mathbb{Z}^d$, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_N \in \mathcal{P} \cap \mathbb{Z}^d$ such that $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_N$,where $N\mathcal{P} = \{N\alpha \mid \alpha \in \mathcal{P}\}$.

Example (Gorenstein Fano polytope)

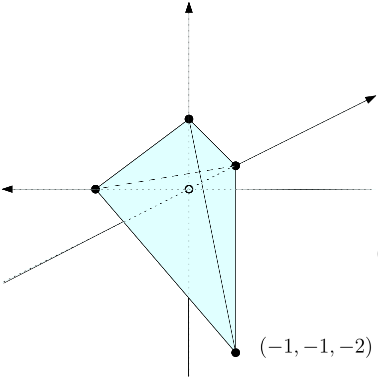


$\mathcal{P}$

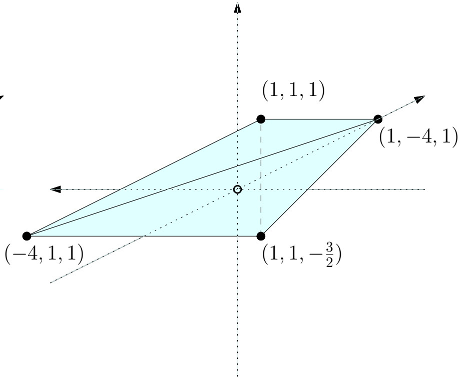


$\mathcal{P}^\vee$

Example (Non-Gorenstein Fano polytope)

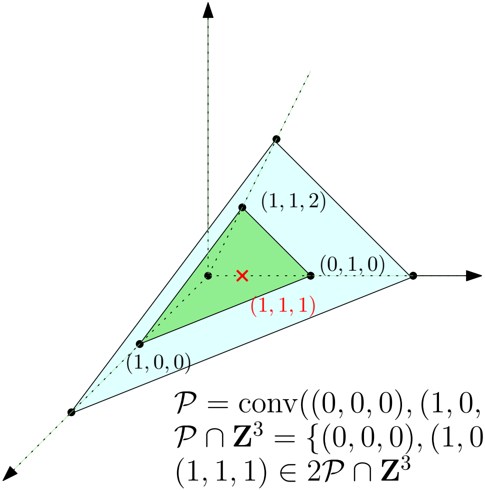


$\mathcal{P}$



$\mathcal{P}^\vee$

Example (Non-normal polytope)



$\mathcal{P} = \mathrm{conv}((0,0,0),(1,0,0),(0,1,0),(1,1,2))$
$\mathcal{P} \cap \mathbf{Z}^3 = \{(0,0,0),(1,0,0),(0,1,0),(1,1,2)\}$
$(1,1,1) \in 2\mathcal{P} \cap \mathbf{Z}^3$

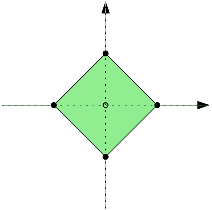
Unimodular equivalence

$\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$: integral convex polytopes of dimension d

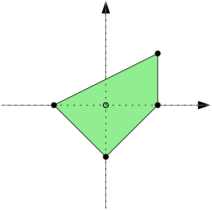
- $\mathcal{P}$ and $\mathcal{Q}$ are unimodularly equivalent

$\stackrel{\text{def}}{\Longleftrightarrow}$ There exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ (i.e., $\det(U) = \pm 1$) and an integer vector $w \in \mathbb{Z}^d$ such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation of $\mathbb{R}^d$ (i.e., $f_U(v) = vU$ for all $v \in \mathbb{R}^d$).

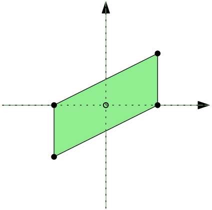
Example



$\mathcal{P}$



$\mathcal{Q}$



$\mathcal{R}$

$\mathcal{P}$ and $\mathcal{Q}$ are not unimodularly equivalent.

$\mathcal{P}$ and $\mathcal{R}$ are unimodularly equivalent.

How many?

Theorem (Lagarias-Ziegler, 1991)

There are only finitely many Gorenstein Fano polytopes up to unimodular equivalence in each dimension.

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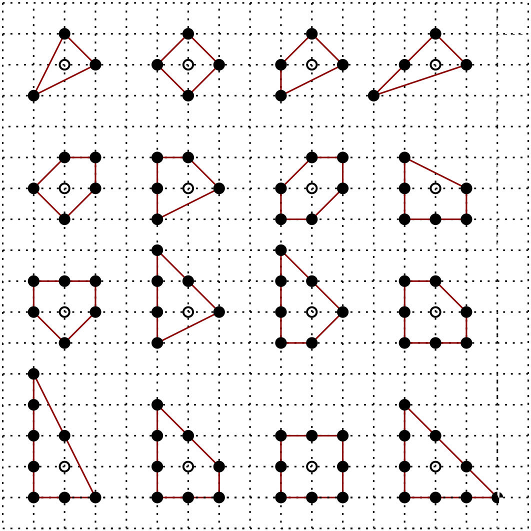
dimension	$\sharp$ of Gorenstein Fano polytopes
1	1
2	16
3	4,319
4	473,800,776
≥ 5	open

Example (Gorenstein Fano polytopes of dimension 2)



Faces of Gorenstein Fano polytopes

Theorem (Haase-Melnikov, 2004)

Every integral convex polytope is unimodularly equivalent to a face of some Gorenstein Fano polytope.

Faces of Gorenstein Fano polytopes

Theorem (Haase-Melnikov, 2004)

Every integral convex polytope is unimodularly equivalent to a face of some Gorenstein Fano polytope.

Question

Is every normal integral convex polytope unimodularly equivalent to a face of some normal Gorenstein Fano polytope?

Partially Ordered SET

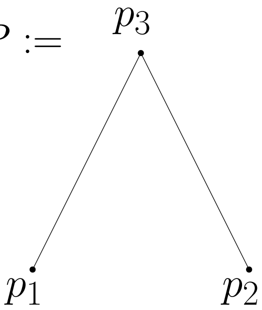
Let $P = \{p_1, \ldots, p_d\}$ be a partially ordered set.

- $I \subset P$: a poset ideal of P
  $\overset{\text{def}}{\Longleftrightarrow}$ $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$.
- $A \subset P$: an antichain of P
  $\overset{\text{def}}{\Longleftrightarrow}$ p_i and p_j belonging to A with $i \neq j$ are incomparable.
- $\sigma = i_1 i_2 \cdots i_d \in S_d$: a linear extension of P
  $\overset{\text{def}}{\Longleftrightarrow}$ $i_a < i_b$ if $p_{i_a} < p_{i_b}$ in P.

We write $\mathcal{J}(P), \mathcal{A}(P)$ and $E(P)$ for the set of poset ideals, antichains and linear extensions of P.

Example

$P :=$



$$\mathcal{J}(P) = \{\emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}, \{p_1, p_2, p_3\}\}$$

$$\mathcal{A}(P) = \{\emptyset, \{p_1\}, \{p_2\}, \{p_3\}, \{p_1, p_2\}\}$$

$$E(P) = \{123, 213\}$$

Order polytopes and Chain polytopes

For each subset $I \subset P$, we define $\rho(I) = \sum_{p_i \in I} \mathbf{e}_i \in \mathbb{R}^d$, where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the canonical unit coordinate vectors of $\mathbb{R}^d$.

Richard Stanley introduced the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ arising from a partially ordered set P.

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Definition

Let $P = \{p_1, \ldots, p_d\}$ be a partially ordered set.

$$\mathcal{O}(P) := \mathrm{conv}(\{\rho(I) \mid I \in \mathcal{J}(P)\}),$$

$$\mathcal{C}(P) := \mathrm{conv}(\{\rho(A) \mid A \in \mathcal{A}(P)\})$$

Example

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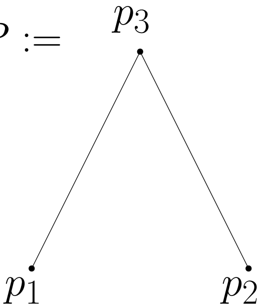

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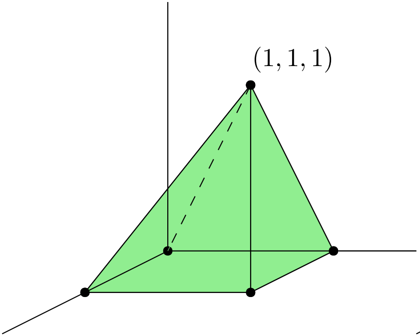
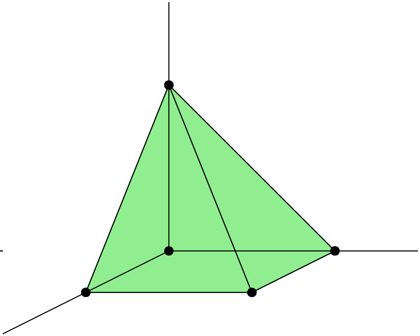


$\mathcal{O}(P)$



$\mathcal{C}(P)$

$$\mathrm{Vol}(\mathcal{O}(P)) = \mathrm{Vol}(\mathcal{C}(P)) = \frac{1}{3}$$

Properties of Two Poset Polytopes

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- $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are integral convex polytopes of dimension d.
- $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are normal.
- $\text{Vol}(\mathcal{O}(P)) = \text{Vol}(\mathcal{C}(P))$.
- $|n\mathcal{O}(P) \cap \mathbb{Z}^d| = |n\mathcal{C}(P) \cap \mathbb{Z}^d|$ for any $n \geq 1$.

Higher dimensional construction

Let $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^d$ be integral convex polytopes of dimension d.
We set

$$\Omega(\mathcal{P}, \mathcal{Q}) = \mathrm{conv}(\mathcal{P} \times \{1\} \cup (-\mathcal{Q}) \times \{-1\}) \subset \mathbb{R}^{d+1},$$

where $-\mathcal{Q} = \{-\alpha | \alpha \in \mathcal{Q}\}$.

Then $\mathcal{P}$ and $-\mathcal{Q}$ are facets of $\Omega(\mathcal{P}, \mathcal{Q})$.

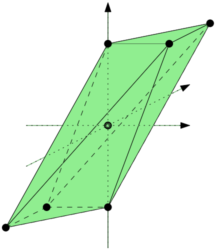
Three types polytope

Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be partially ordered sets.
We consider the following polytopes

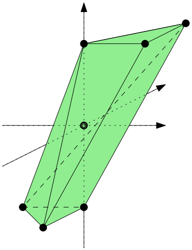
$$\Omega(\mathcal{O}(P), \mathcal{O}(Q)), \Omega(\mathcal{O}(P), \mathcal{C}(Q)), \Omega(\mathcal{C}(P), \mathcal{C}(Q)).$$

We want to know when these polytopes are normal and Gorenstein Fano.

Example



$\Omega(\mathcal{O}(P), \mathcal{O}(Q))$



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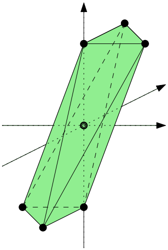
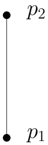


$\Omega(\mathcal{C}(P), \mathcal{C}(Q))$

$P =$ 

$Q =$ 

When are these polytopes Gorenstein Fano?

Theorem (Hibi-T)

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Let $P = \{p_1, \ldots, p_d\}$ *and* $Q = \{q_1, \ldots, q_d\}$ *be partially ordered sets.*

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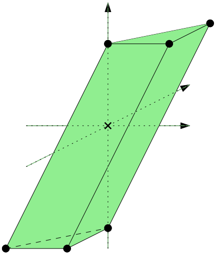
(iii) $\Omega(\mathcal{C}(P), \mathcal{C}(Q))$ *is always normal Gorenstein Fano.*

Example



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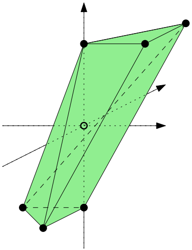
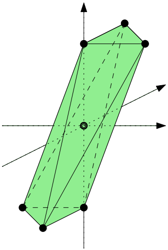


$\Omega(\mathcal{C}(P), \mathcal{O}(Q))$



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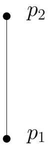
$P =$ 

$Q =$ 

How to prove?

We set

$\mathcal{P} \subset \mathbb{R}^d$: An integral convex polytope of dimension d.

$S = K[x_1, \ldots, x_d, t]$: polynomial ring.

$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$.

$K[\mathcal{P}] = K[\{x^\alpha t : \alpha \in \mathcal{P} \cap \mathbb{Z}^d\}] \subset S$: The toric ring of $\mathcal{P}$.

$\phi : T = K[\{z_\alpha : \alpha \in \mathcal{P} \cap \mathbb{Z}^d\}] \to K[\mathcal{P}]$ $(z_\alpha \mapsto x^\alpha t)$.

$I_{\mathcal{P}} = \ker \phi$: The toric ideal of $\mathcal{P}$.

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$I_{\mathcal{P}} = \ker \phi$: The toric ideal of $\mathcal{P}$.

Lemma (Hibi-Matsuda-Ohsugi-Shibata, 2015)

Let $\mathcal{P} \subset \mathbb{R}^d$ be a Fano polytope of dimension d. Suppose that $\sum_{\alpha \in \mathcal{P} \cap \mathbb{Z}^d} \mathbb{Z}(\alpha, 1) = \mathbb{Z}^{d+1}$ and there exists a reverse lexicographic order $<_{rev}$ on T such that

- $z_{(0,\dots,0)}$ *is smallest.*
- $in_{<_{rev}}(I_{\mathcal{P}})$ *is squarefree.*

Then $\mathcal{P}$ is normal Gorenstein Fano.

Gröbner bases

Set $K[\mathcal{O}\mathcal{O}] = K[\{x_I\}_{I \in \mathcal{J}(P)} \cup \{y_J\}_{J \in \mathcal{J}(Q)} \cup \{z\}]$.
$\pi_{\mathcal{O}\mathcal{O}} : K[\mathcal{O}\mathcal{O}] \to K[\Omega(\mathcal{O}(P), \mathcal{O}(Q))]$ by setting

- $\pi_{\mathcal{O}\mathcal{O}}(x_I) = \mathbf{t}^{\rho(I \cup \{d+1\})} s$,
- $\pi_{\mathcal{O}\mathcal{O}}(y_J) = \mathbf{t}^{-\rho(J \cup \{d+1\})} s$,
- $\pi_{\mathcal{O}\mathcal{O}}(z) = s$.

Let $<_{\mathcal{O}\mathcal{O}}$ denote a reverse lexicographic order on $K[\mathcal{O}\mathcal{O}]$ satisfying

- $z <_{\mathcal{O}\mathcal{O}} y_J <_{\mathcal{O}\mathcal{O}} x_I$;
- $x_{I'} <_{\mathcal{O}\mathcal{O}} x_I$ if $I' \subset I$;
- $y_{J'} <_{\mathcal{O}\mathcal{O}} y_J$ if $J' \subset J$,

and $\mathcal{G}_{\mathcal{O}\mathcal{O}} \subset K[\mathcal{O}\mathcal{O}]$ the set of the following binomials:

(i) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$ (I and I' are incomparable in $\mathcal{J}(P)$)
(ii) $y_J y_{J'} - y_{J \cup J'} y_{J \cap J}$ (J and J' are incomparable in $\mathcal{J}(Q)$)
(iii) $x_I y_J - x_{I \setminus \{p_i\}} y_{J \setminus \{q_i\}}$ (p_i, q_i are maximal elements of I, J)
(iv) $x_\emptyset y_\emptyset - z^2$,

Gröbner bases

Proposition

If P and Q possess a common linear extension, then $\Omega(\mathcal{O}(P), \mathcal{O}(Q))$ is Fano and $\mathcal{G}_{OO}$ is a Gröbner basis of $I_{\Omega(\mathcal{O}(P),\mathcal{O}(Q))}$ with respect to $<_{OO}$.

Similarly, we can show the case of $\Omega(\mathcal{O}(P), \mathcal{C}(Q))$ and $\Omega(\mathcal{C}(P), \mathcal{C}(Q))$.
Hence the assersion of theorem follows.

Questions

Question

For any $(0,1)$-polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d, does there exist $(0,1)$-polytope $\mathcal{Q} \subset \mathbb{R}^d$ of dimension d such that $\Omega(\mathcal{P}, \mathcal{Q})$ is a Gorenstein Fano polytope?

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Question

For any normal $(0,1)$-polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d, does there exist normal $(0,1)$-polytope $\mathcal{Q} \subset \mathbb{R}^d$ of dimension d such that $\Omega(\mathcal{P}, \mathcal{Q})$ is a normal Gorenstein Fano polytope?

Example

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Let $\mathcal{P} \subset \mathbb{R}^9$ be the $(0,1)$-polytope of dimension 9 whose vertices are followings:

$$e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4 + e_5, e_1 + e_5, e_1 + e_6, e_1 + e_7,$$

$$e_2 + e_7, e_2 + e_8, e_3 + e_8, e_3 + e_9, e_4 + e_9, e_4, e_5, e_5 + e_6.$$

Then $\mathcal{P}$ is normal. Moreover, $\Omega(\mathcal{P}, \mathcal{P})$ is Gorenstein Fano, but it is not normal.

Ehrhart polynomial

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d.
We set

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^d|, n = 1, 2, \ldots.$$

$i(\mathcal{P}, n)$ is a polynomial in n of degree d.
We call $i(\mathcal{P}, n)$ the Ehrhart polynomial of $\mathcal{P}$.
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The following properties are known:

(i) The constant of $i(\mathcal{P}, n)$ equals 1;

(ii) The leading coefficient of $i(\mathcal{P}, n)$ equals $\mathrm{vol}(\mathcal{P})$;

Remark

$i(\mathcal{O}(P), n) = i(\mathcal{C}(P), n)$ for any partially ordered set.
Hence $\mathrm{vol}(\mathcal{O}(P)) = \mathrm{vol}(\mathcal{C}(P))$.

Hilbert polynomial

In fact,

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- Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d. If $\mathcal{P}$ is normal, then the Ehrhart polynomial of $\mathcal{P}$ is equal to the Hilbert function of the toric ring $K[\mathcal{P}]$.

Hilbert polynomial

In fact,

- Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d. If $\mathcal{P}$ is normal, then the Ehrhart polynomial of $\mathcal{P}$ is equal to the Hilbert function of the toric ring $K[\mathcal{P}]$.
- Let S be a polnomial ring and $I \subset S$ be a graded ideal of S. Let $<$ be a monomial order on S. Then S/I and $S/\text{in}_<(I)$ have the same Hilbert function.

Toric rings

Proposition

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Toric rings

Proposition

Let $P = \{p_1, \ldots, p_d\}$ *and* $Q = \{q_1, \ldots, q_d\}$ *be partially ordered sets.*

(i) $K[\mathcal{OC}]/\mathrm{in}_{<\mathcal{OC}}(I_{\Omega(\mathcal{O}(P),\mathcal{C}(Q))}) \cong K[\mathcal{CC}]/\mathrm{in}_{<\mathcal{CC}}(I_{\Omega(\mathcal{C}(P),\mathcal{C}(Q))})$.

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(ii) *If* P *and* Q *possess a common linear extension, then*
$K[\mathcal{OO}]/\mathrm{in}_{<_{\mathcal{OO}}}(I_{\Omega(\mathcal{O}(P),\mathcal{O}(Q))})$
$\cong K[\mathcal{OC}]/\mathrm{in}_{<_{\mathcal{OC}}}(I_{\Omega(\mathcal{O}(P),\mathcal{C}(Q))})$
$\cong K[\mathcal{CC}]/\mathrm{in}_{<_{\mathcal{CC}}}(I_{\Omega(\mathcal{C}(P),\mathcal{C}(Q))})$.

Ehrhart polynomials of three types polytopes

Theorem (Hibi-T)

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(i) $\Omega(\mathcal{O}(P), \mathcal{C}(Q))$, $\Omega(\mathcal{C}(P), \mathcal{C}(Q))$ *have the same Ehrhart polynomial. In particular, these polytopes have the same volume.*

Ehrhart polynomials of three types polytopes

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(i) $\Omega(\mathcal{O}(P), \mathcal{C}(Q))$, $\Omega(\mathcal{C}(P), \mathcal{C}(Q))$ *have the same Ehrhart polynomial. In particular, these polytopes have the same volume.*

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formula of volume

Theorem (Stanley, 1986)

Let $P = \{p_1, \ldots, p_d\}$ be a partially ordered set. Then we have

$$\mathsf{Vol}(\mathcal{O}(P)) = \mathsf{Vol}(\mathcal{C}(P)) = \frac{\sharp E(P)}{d!}$$

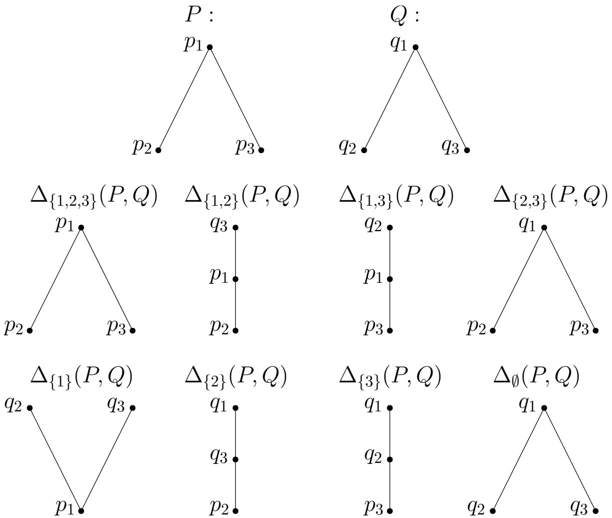
formula of volume

- For partially ordered sets P and Q with $P \cap Q = \emptyset$, the ordinal sum of P and Q is the partially ordered set $P \oplus Q$ on the union $P \cup Q$ such that $s \leq t$ in $P \oplus Q$ if and only if
  (a) $s, t \in P$ and $s \leq t$ in P, or
  (b) $s, t \in Q$ and $s \leq t$ in Q, or
  (c) $s \in P$ and $t \in Q$.

Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be partially ordered sets.

- For $W \subset [d] = \{1, \ldots, d\}$, We write P_W for the partially ordered set $\{p_i \mid i \in W\}$ such that $p_i \leq p_j$ in P_W if and only if $p_i \leq p_j$ in P.
- For $W \subset [d]$, we write $\Delta_W(P, Q) = P_W \oplus Q_{\overline{W}}$, where $\overline{W} = [d] \setminus W$.

Example


$P:$
p_1
p_2
p_3
$Q:$
q_1
q_2
q_3
$\Delta_{\{1,2,3\}}(P,Q)$
p_1
p_2
p_3
$\Delta_{\{1,2\}}(P,Q)$
q_3
p_1
p_2
$\Delta_{\{1,3\}}(P,Q)$
q_2
p_1
p_3
$\Delta_{\{2,3\}}(P,Q)$
q_1
p_2
p_3
$\Delta_{\{1\}}(P,Q)$
q_2
q_3
p_1
$\Delta_{\{2\}}(P,Q)$
q_1
q_3
p_2
$\Delta_{\{3\}}(P,Q)$
q_1
q_2
p_3
$\Delta_{\emptyset}(P,Q)$
q_1
q_2
q_3

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Theorem (Hibi-T)

Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be partially ordered sets, and set $P' = \{p_{d+1}\} \oplus P$ and $Q' = \{q_{d+1}\} \oplus Q$. If P and Q have a same linear extension, then we have

$$\mathrm{vol}(\Omega(\mathcal{O}(P), \mathcal{O}(Q))) = \sum_{W \subset [d+1]} \frac{\sharp E(\Delta_W(P', Q'))}{(d+1)!}.$$