Structure and Classifications of Lattice Polytopes

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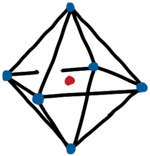
Technische Universität Darmstadt

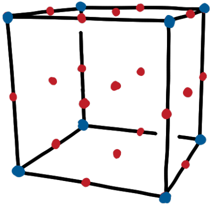
(Classifications of) Lattice Polytopes

TECHNISCHE UNIVERSITÄT DARMSTADT

- lattice polytope: convex hull of finitely many points in $\mathbb{Z}^d$

$$Q = \operatorname{conv}(v_1, \dots, v_n), \qquad v_i \in \mathbb{Z}^d$$





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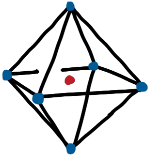
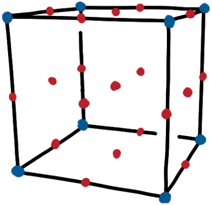
  - by dimension
  - number of (interior) lattice points
  - structural properties:
    - many vertices, h^*-vector, ...





polymake Discrete Optimization

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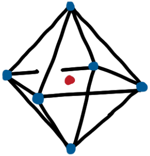

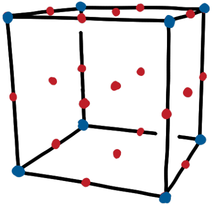

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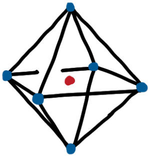

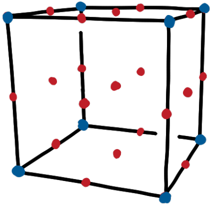

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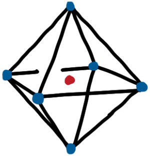

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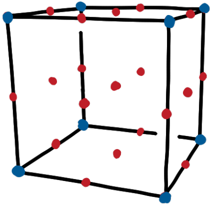

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  - computational
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  - structural
    - standard polytope constructions
    - projections, liftings
    - common properties
    - . . .

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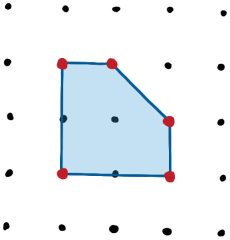

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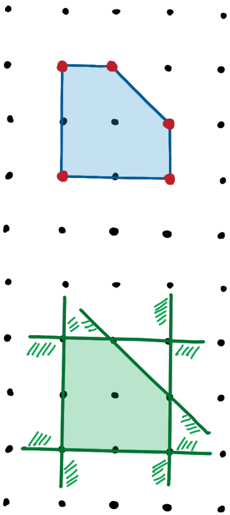

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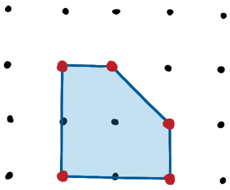

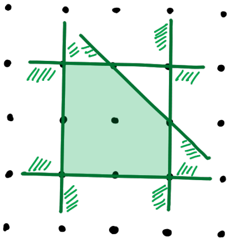

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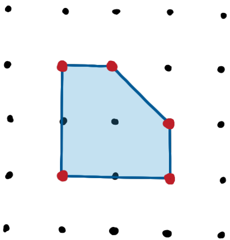

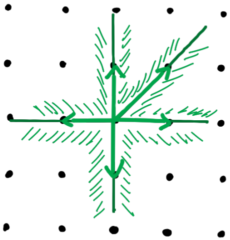

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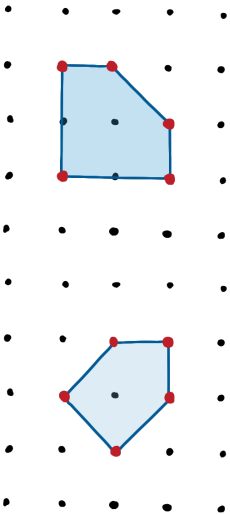
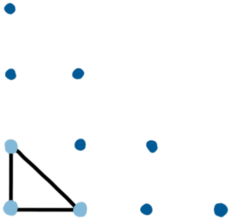
- polar (dual) polytope:

$$Q^\vee = \{ x \mid \langle x, v \rangle \leq 1 \quad \forall v \in Q \}$$



Lattice Polytopes

- ▷ Ehrhart polynomial $\operatorname{ehr}_P(k) := |k \cdot P \cap \mathbb{Z}^d|$ polynomial of degree d



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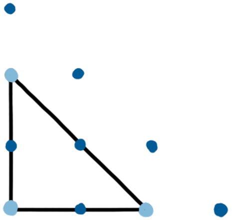

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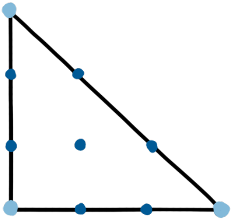

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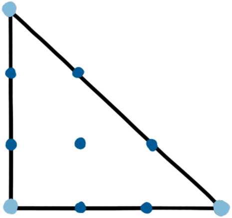

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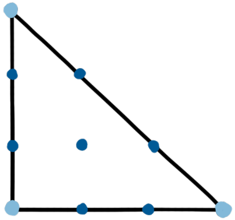

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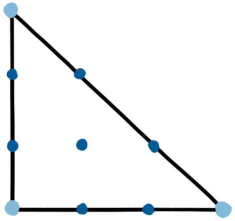

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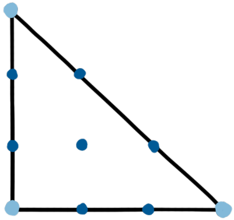

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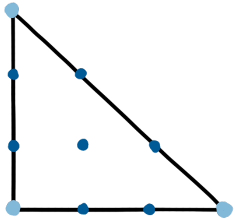

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- toric dictionary: properties of variety correpond to properties of polytope

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$d, m \geq 1$. Then there are, up to lattice equivalence, only finitely many d-dimensional lattice polytopes with m interior lattice points.

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$\longrightarrow$ different approaches for classifications of *empty polytopes* and those with *interior lattice points*.

Empty Lattice Polytopes

- $h^*_d = |\operatorname{int} P \cap \mathbb{Z}^d|$, so $h^*_d = 0$ iff P is empty

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- $d + 1 - \max(k \mid h^*_k \neq 0) = \min(k \mid \operatorname{int}(kP) \cap \mathbb{Z}^d \neq \emptyset)$

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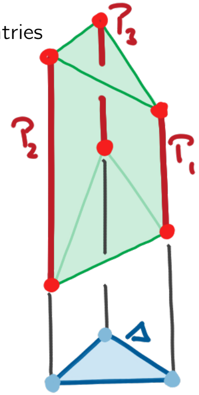
$$P = \operatorname{conv}(P \times \{e_i\} \mid i = 1, \ldots, k)$$

for $k \geq 2$ polytopes P_i and the unit vectors $e_i \in \mathbb{Z}^k$

▷ equivalently, P (lattice) projects onto a unimodular simplex of dimension at least 1



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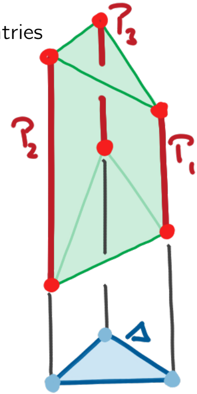

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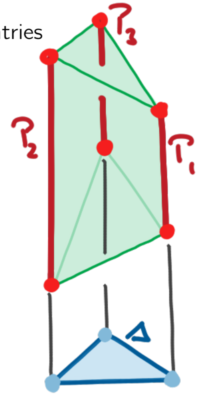

- intermediate zeros in h^* :

  ▷ $h^*_1 \;=\; |P \cap \mathbb{Z}^d| \,-\, d \,-\, 1,$

  ▷ $h^*_1 = 0 \quad\Longrightarrow\quad P$ is an empty simplex

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Polytopes with one interior lattice point

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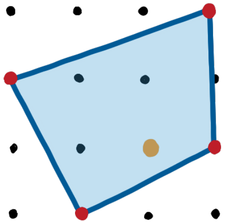

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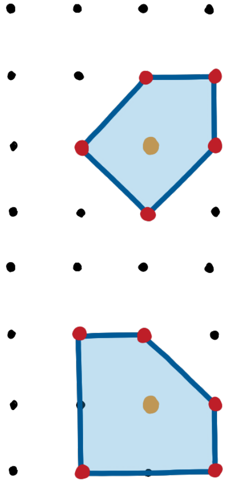

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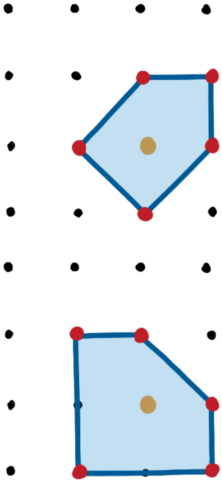

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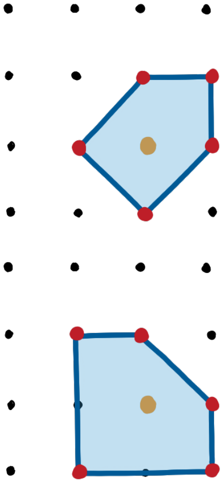

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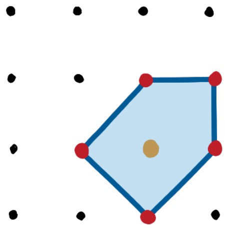

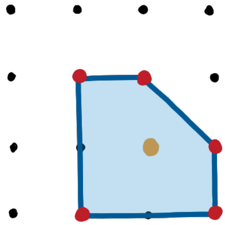

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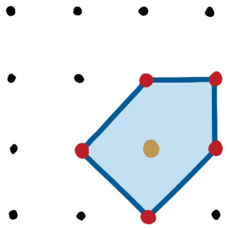

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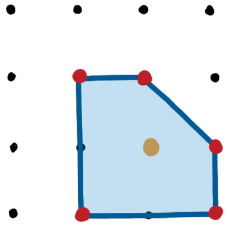

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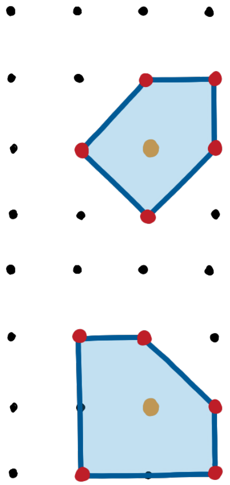

Computational Classifications

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Theorem (Hensley; Lagarias & Ziegler)

$d, m \geq 1$. Then there are, up to lattice equivalence, only finitely many d-dimensional lattice polytopes with m interior lattice points.

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- canonical/terminal polytopes can be *grown* from minimal ones by adding vertices
- smooth reflexive polytopes *cannot* be grown from minimal ones
  $\longrightarrow$ construction depends on notion of *special facet* and a total order on potential vertices

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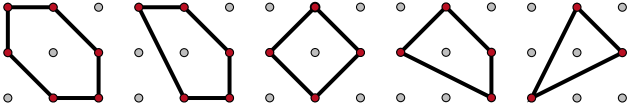
  - $d = 2$: 

P_6 P_5 P_{4a} P_{4b} P_3

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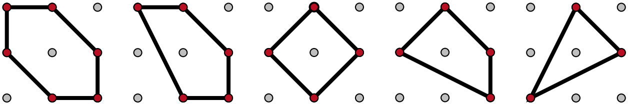

P_6 P_5 P_{4a} P_{4b} P_3

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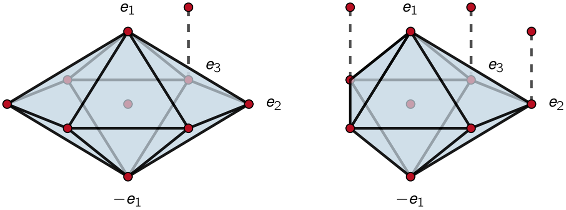

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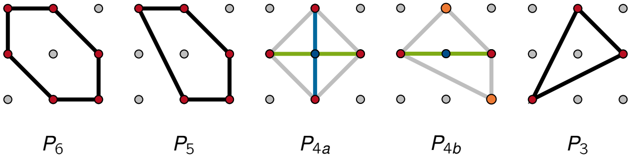
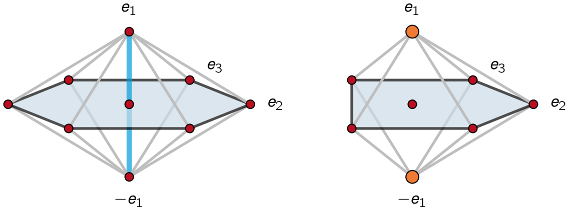
  - $d = 2$: 

  - $d = 3$: 

General Constructions

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- P, P' polytopes containing 0 in their interior.
- direct sum $P \oplus P' := \operatorname{conv}(P \times \{0\} \cup \{0\} \times P')$
- bipyramid $\operatorname{bipyr}(P) := \operatorname{conv}(\{0\} \times P \cup \{e_1, -e_1\})$
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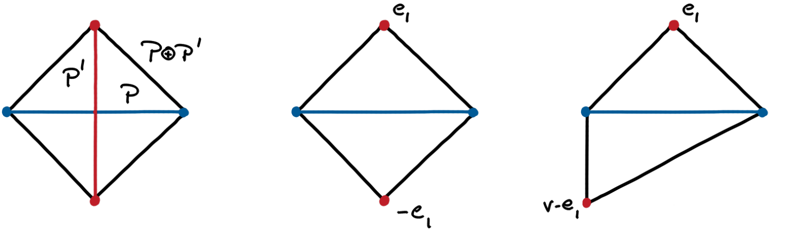

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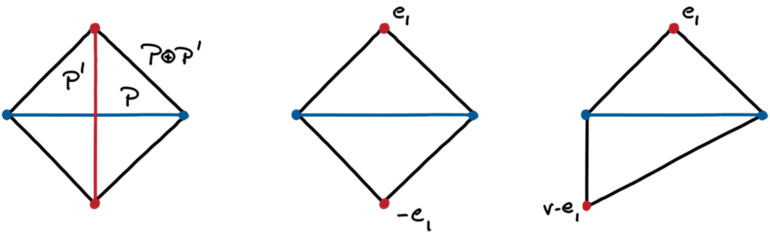

- **Proposition** constructions preserve simplicial/terminal/reflexive

Basic Examples

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(1) regular cross polytope: $C(d) \;:=\; \mathrm{conv}(\,\pm \mathrm{e}_i \;\mid\; 1 \leq i \leq d\,) \;\subset\; \mathbb{R}^d$

(2) pseudo-Del Pezzo polytope: $D'(d) \;:=\; \mathrm{conv}\,(\,C(d) \;\cup\; \{\mathbb{1}\}\,) \;\subset\; \mathbb{R}^d$

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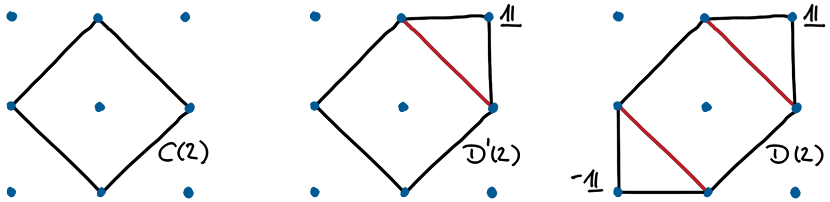

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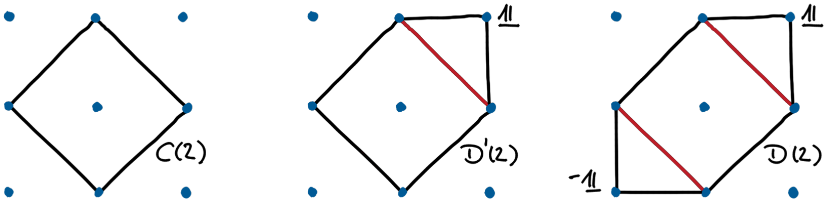

▶ Theorem [Voskresenskii&Klyachko, Ewald, Nill]

P simplicial, terminal, and reflexive with antipodal pair of facets

$\Longrightarrow$ P is direct sum of a centrally symmetric cross polytope, (2), and (3)

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Many Vertices

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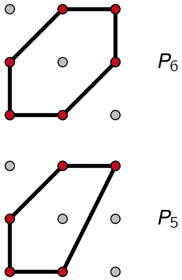
- $f_0 = 3d$: **(a)** $P_6^{\oplus d/2}$



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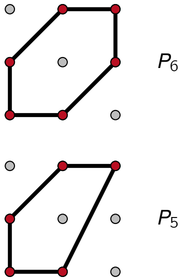

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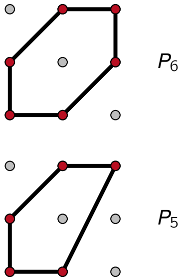

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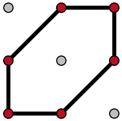
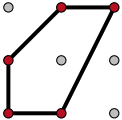


P_6



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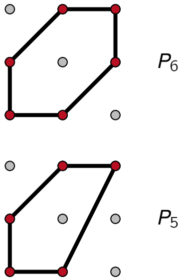

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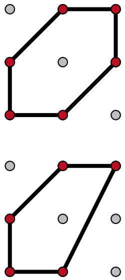

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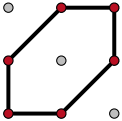

P_6

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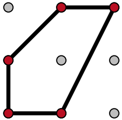

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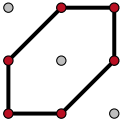
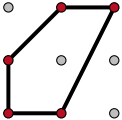
  (e) $D(4) \oplus P_6^{\oplus d/2-2}$
  (f) proper or skew bipyramid over **(b)** or **(c)**
  (g) double proper or skew bipyramid over **(a)**
- Theorem There are no other cases. [Assarf, Joswig, P]





η-vectors

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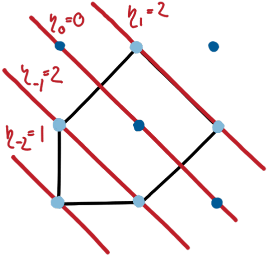
- P simplicial, terminal, and reflexive d-polytope,
  - F a facet of P with normal u_F

    $\longrightarrow$ given by primitive facet normal u_F

    $$F = \{x \in P \mid \langle u_F, x \rangle = 1\}$$



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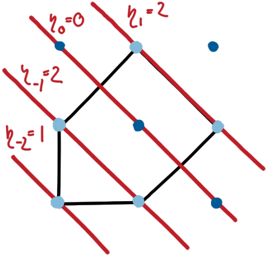
    $\longrightarrow$ u_F induces grading on $V(P)$ by distance from F



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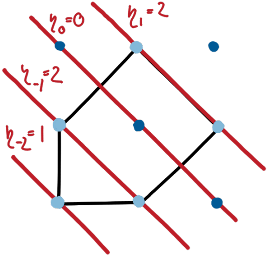
    $$\eta_i := |\{x \in V(P) \mid \langle u, x \rangle = i\}|$$



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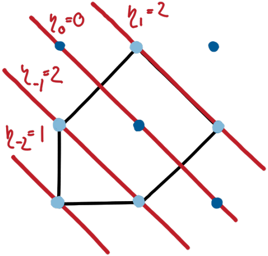
$\longrightarrow$ Partition vertex set

$$V(P) := V(F) \cup V(F,0) \cup V(F,-1) \cup \dots$$



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η-vectors

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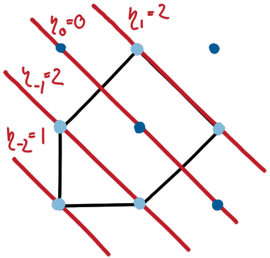

- F is a special facet

$$:\Longleftrightarrow \quad v_P \; := \; \sum_{v \in V(P)} v \; \in \; \mathrm{cone}(F)$$

Many Vertices

- Fix special facet F,
  - vertices $\{u_i\}$, dual basis $\{\hat{u}_i\}$
  - $V(P) \; := \; V(F) \; \cup \; V(F, 0) \; \cup \; V(F, -1) \; \cup \ldots$

Many Vertices

TECHNISCHE UNIVERSITÄT DARMSTADT

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- Proposition [Øbro]
  - Coordinates of vertices are bounded in dual basis
    - $\mathrm{x} \in V(F,k) \quad \Longrightarrow \quad \langle \hat{\mathrm{u}}_i, \mathrm{x} \rangle \;\geq\; k - 1$

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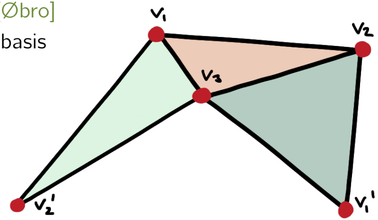

Many Vertices

TECHNISCHE UNIVERSITÄT DARMSTADT

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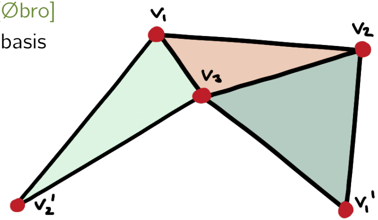

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Many Vertices

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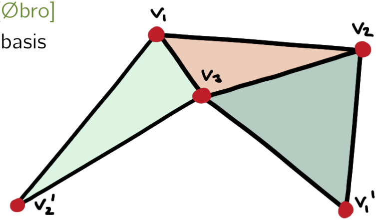
    - Vertices in $V(F,0)$ are on facets adjacent to F



Many Vertices

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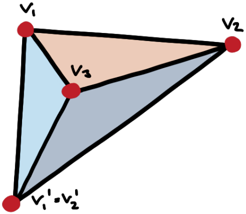

Many Vertices

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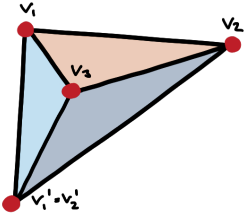

- **Proposition** $\eta_0 \;\leq\; d$ [Nill]

Many Vertices

TECHNISCHE UNIVERSITÄT DARMSTADT

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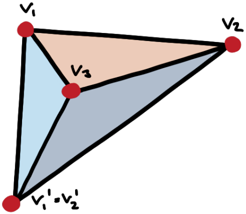
- Proposition $\eta_0 \;\leq\; d$ [Nill]
- Proposition $\eta_0 \;\geq d-1 \quad\Longrightarrow\quad u_1,\ldots,u_d$ are lattice basis



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Many Vertices

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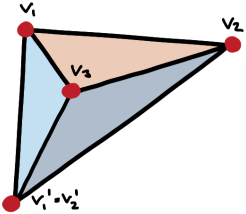

- Proposition $\eta_0 \leq d$ [Nill]
- Proposition $\eta_0 \geq d-1 \implies u_1, \ldots, u_d$ are lattice basis
- Theorem $f_0 := |V(P)| \leq 3d$ [Casagrande; Øbro]

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Many Vertices

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- Fix special facet F,
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- **Proposition** [Øbro]
  - Coordinates of vertices are bounded in dual basis
    - $x \in V(F,k) \implies \langle \hat{u}_i, x\rangle \geq k-1$
    - **equality**: $V(F) - \{u_i\} + \{x\}$ is facet
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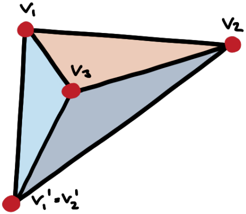

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- **Theorem** $f_0 := |V(P)| \leq 3d$ [Casagrande; Øbro]

  proof: $0 \leq \langle u_F, \sum_{v\in V(P)} v \rangle = \eta_1 + 0\cdot\eta_0 + (-1)\cdot\eta_{-1} + (-2)\cdot\eta_{-2} + \cdots$

  $= d + 0 - \cdots$

Many Vertices

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- classify η-vectors for a special facet F
- $\mathrm{v}_P := \sum_{\mathrm{v} \in \mathrm{V}(P)} \mathrm{v}$
- ℓ: height of v_P above F

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Many Vertices

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ℓ	2	1	1	0	0	0	0
η_1	d	d	d	d	d	d	d
η_0	d	d	$d-1$	d	d	$d-1$	$d-2$
η_{-1}	$d-2$	$d-3$	$d-1$	$d-3$	$d-4$	$d-2$	d
η_{-2}	0	1	0	0	2	1	0
η_{-3}	0	0	0	1	0	0	0
	(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**

(d)–(g): all facets special

Many Vertices

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- consider cases separately

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Many Vertices

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▷ *e.g.*, ▷ all η^F of type **(g)** $\Longrightarrow$ polytope is centrally symmetric

Many Vertices

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  - ▷ **(d)** does not occur $\longleftarrow$ look at adjacent facet

Many Vertices

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- show that polytopes are either
  - ▷ direct sum of P_6 with $(d-2)$-polytope
  - ▷ (skew) bipyramid over $(d-1)$-polytope

Many Vertices

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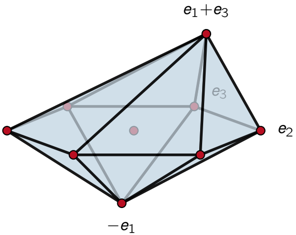
- exact types via induction

Can we continue?

- $f_0 = 3d - 3$?
  - R := skew bipyramid over P_6
    $\longrightarrow$ 8 vertices and 12 facets
  - P := $R^{\oplus 3}$
    $\longrightarrow$ $3 \cdot 8 = 3 \cdot 9 - 3$ vertices in dimension $d = 9$
  - P is not a (skew) bipyramid over a sum of P_5 and P_6



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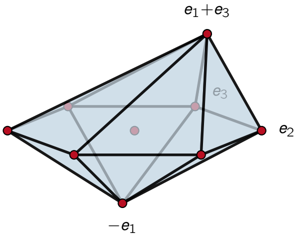

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- Theorem [Assarf, Joswig, P]

  P terminal, simplicial, reflexive d-polytope with $3d - 2$ vertices

  Then P is
  - $P_5^2 \oplus P_6^{\oplus d/2-2}$, or
  - $D(4) \oplus P_6^{\oplus d/2-2}$, or
  - (double) proper/skew bipyramid over $P_6^{\oplus k}$ for suitable k

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Can we continue?

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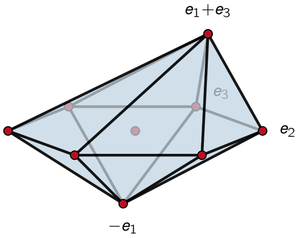

- Theorem [Assarf, Joswig, P]

  P terminal, simplicial, reflexive d-polytope with $3d - 2$ vertices
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Can we continue?

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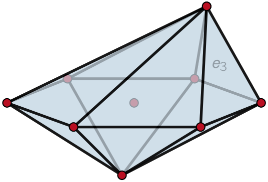

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- Theorem [Assarf, Joswig, P]
  P terminal, simplicial, reflexive d-polytope with $3d - 2$ vertices
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- Conjecture [Assarf, Joswig, P]
  P smooth Fano d-polytope with $3d - k$ vertices, $k \leq d/3$
  Then $P = Q \oplus P_6^l$ for $\dim Q \leq 3k$ and appropriate l.

Can we continue?

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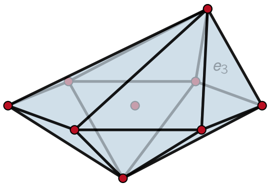

- Theorem [Assarf, Joswig, P]

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- Conjecture [Assarf, Joswig, P]

  P smooth Fano d-polytope with $3d - k$ vertices, $k \leq d/3$
  Then $P = Q \oplus P_6^l$ for dim $Q \leq 3k$ and appropriate l.

- weak version of conjecture is true:
- Theorem [Assarf, Nill]

  For sufficiently large d, v a smooth Fano d-polytope with v vertices has a P_6-factor.

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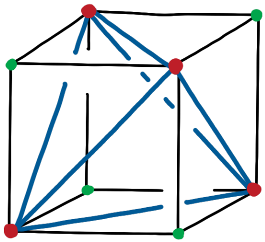
$\longrightarrow$ S is lattice simplex of volume and facet width 2



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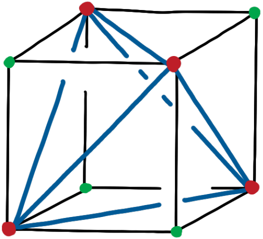

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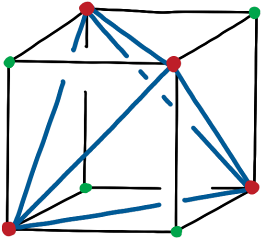

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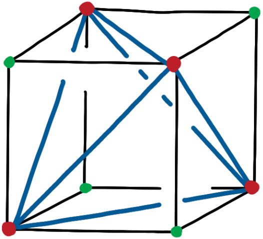

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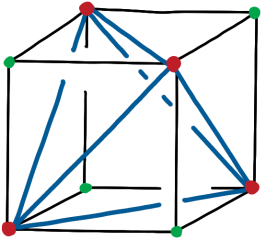

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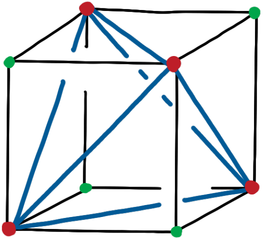

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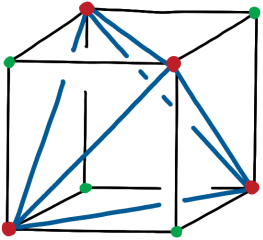

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- open case: simplicial, terminal, reflexive polytopes with $3d - 3$ vertices

polymake

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- polyDB: database extension for polymake
  - direct access from polymake
  - independent access/access from other software possible
  - web based interface (planned)

  beta version, developed by: Silke Horn (iteratec), P.
  available at github.org/solros/poly_db, GPL licensed