Ehrhart theory for spanning lattice polytopes

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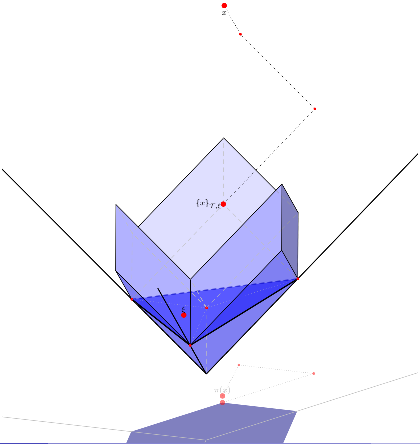

Overview

- Motivation
- Theorem and Applications
- Proof for simplices
- Half-open triangulations
- Sketch of proof

Notation

$P \subset \mathbb{R}^d$ d-dimensional lattice polytope

$$\sum_{k=0}^{\infty} |(kP) \cap \mathbb{Z}^d| t^k = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

$h_P^* = (h_0^*, \ldots, h_s^*, 0 \ldots, 0)$ with $h_s^* \neq 0$,
$s = \deg(P)$

Motivation: low degree

$$s = 0 \quad \Longleftrightarrow \quad P \cong \Delta_d \text{ unimodular simplex}$$

$s = 1$: classification known (Batyrev, N. '07)

Motivation: vanishing of h^*-coefficients

Question:

$$h_i^* = 0 \quad \overrightarrow{\overleftarrow{}} \quad ?$$

Motivation: $i = 1$

$$h_1^* = 0 \quad \Longleftrightarrow \quad |P \cap \mathbb{Z}^{d+1}| = d + 1 \quad \Longleftrightarrow \quad P \text{ empty simplex}$$

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Let us give alternative formulation.

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Definition

Γ_P **sublattice** of $\mathbb{Z}^{d+1}$ generated by $(P \times \{1\}) \cap \mathbb{Z}^{d+1}$.

$\tilde{\mathbf{P}} \times \{1\}$ is lattice polytope *with respect to* Γ_P having the vertices of P .

In order to write $\tilde{P}$ as lattice polytope of $\mathbb{Z}^d$ one has to pick affine lattice basis of the affine sublattice of $\mathbb{Z}^d$ of integer affine combinations of $P \cap \mathbb{Z}^d$.

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Alternative criterion:

$$h_1^* = 0 \quad \Longleftrightarrow \quad \deg(\tilde{P}) = 0$$

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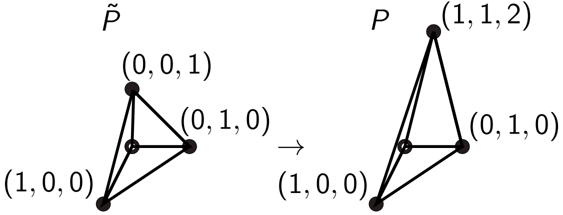
Example:



Motivation: $i = 2$

$h_1^* = 0 \iff \deg(\tilde{P}) = 0$

Proposition (Blekherman, Smith, Velasco '13)

$h_2^* = 0 \iff \deg(\tilde{P}) \leq 1$ and P is 2-IDP

i-**IDP:** every lattice point in iP is a sum of i lattice points in P.

Is there a generalization to $i \geq 3$?

The main theorem

P is **spanning** if and only if $\Gamma_P = \mathbb{Z}^{d+1}$.

Any lattice polytope is spanning after a change of lattice

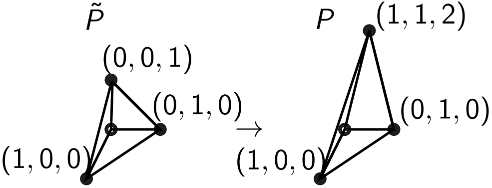
$P \times \{1\}$ lattice polytope (with respect to $\mathbb{Z}^{d+1}$).

$\rightsquigarrow$

$\tilde{P} \times \{1\}$ **spanning** lattice polytope (with respect to Γ_P) with

- same lattice points ($\tilde{h}^*_1 = h^*_1$)
- $\tilde{h}^*_i \leq h^*_i$

Example:



$\tilde{h}^* = (1, 0, 0, 0)$ $\quad$ $h^* = (1, 0, 1, 0)$

The main theorem

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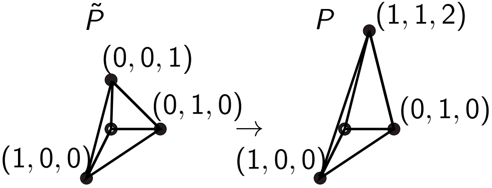

$\tilde{h}^* = (1, 0, 0, 0)$ $\qquad$ $h^* = (1, 0, 1, 0)$

Theorem A (Hofscheier, Katthän, N. '16)

$$P \text{ spanning} \implies h^*_0, \ldots, h^*_s \neq 0 \text{ (no gaps)}$$

Back to our motivation: $i \geq 3$

Proposition (Blekherman, Smith, Velasco '13)

$h_2^* = 0 \iff \deg(\tilde{P}) \leq 1$ and P is 2-IDP

Proposition/Corollary to Theorem A

$\deg(\tilde{P}) \leq i - 1$ and P is i-IDP $\implies h_i^* = 0 \implies \deg(\tilde{P}) \leq i - 1$

Careful: $h_i^* = 0 \overset{i \geq 3}{\not\Longrightarrow}$ i-IDP.

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Geometric consequence:

Corollary to Theorem A and (Haase, N., Payne '09)

$h_i^* = 0$ and $d \geq 20(i-1)^2$
$\implies d \geq 20(\deg(\tilde{P}))^2$
$\implies \tilde{P}$ is a *Cayley polytope*
$\implies$ all lattice points of P lie on two parallel faces.

Relation to unimodality conjecture

Conjecture (Stanley '89; Ohsugi, Hibi '05; Schepers, Van Langenhoven '11)

$$P \text{ IDP (integrally-closed)} \quad \Longrightarrow \quad h^* \text{ is unimodal}$$

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$$P \text{ spanning} \overset{d \geq 5}{\not\Longrightarrow} h^* \text{ is unimodal}$$

Alternative algebraic proof when P

P IDP $\Longrightarrow$ h^* is M-sequence $\Longrightarrow$ no gaps

($h_i^* = \dim_{\mathbb{C}} R_i$ for R standard-graded $\mathbb{C}$-algebra)

Polyhedral consequence of Eisenbud-Goto

Eisenbud-Goto-Conjecture

$\mathbf{k}$ alg. closed, I homogeneous prime ideal in $S := \mathbf{k}[x_1, \ldots, x_n]$. Then

$$\mathrm{reg}(S/I) + \mathrm{codim}(S/I) \leq \deg(S/I).$$

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Eisenbud-Goto-Conjecture is wrong (?) (McCullough, Peeva '16 (?))

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For P **spanning** (here, S/I is **k**-algebra generated by $(P \times \{1\}) \cap \mathbb{Z}^{d+1}$):

$$\deg(S/I) = d!\mathrm{vol}(P) = h^*_0 + h^*_1 + h^*_2 + \cdots + h^*_s$$
$$\mathrm{codim}(S/I) = h^*_1$$
$$\mathrm{reg}(S/I) \geq s$$

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Corollary to Theorem A

$$P \text{ spanning} \quad \Longrightarrow \quad s + h_1^* \leq h_0^* + \cdots + h_s^*$$

Proof: r.h.s. $\iff s - 1 \leq h_2^* + \cdots + h_s^* \iff$
$1 + \cdots + 1 \leq h_2^* + \cdots + h_s^* \quad \square$

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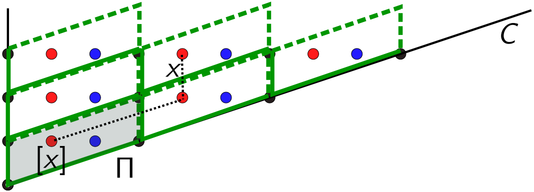
Needs spanning: otherwise $h^* = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$ possible!

Proof idea for spanning simplices

Any coset in $\mathbb{Z}^{d+1}$ modulo lattice spanned by vertices has **unique representative** in $\Pi \cap \mathbb{Z}^{d+1}$:

$$x \in \mathbb{Z}^{d+1} \quad \rightsquigarrow \quad [x] \in \Pi \cap \mathbb{Z}^{d+1}.$$



Proof idea for spanning simplices

Let $x \in \Pi \cap \mathbb{Z}^{d+1}$ with $\mathrm{height}(x) = s$.
P spanning $\Longrightarrow$ exist lattice points in $P \times \{1\}$ such that

$$x = w_1 + \cdots - u_1 - \cdots$$

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$\Pi \cap \mathbb{Z}^{d+1}$ is finite abelian group, so exists $k_j \in \mathbb{N}_{\geq 1}$ such that

$$k_j[u_j] = [0], \text{ so } [-u_j] = (k_j - 1)[u_j].$$

$$x = [x] = [w_1 + \cdots + w_k]$$

Proof idea for spanning simplices

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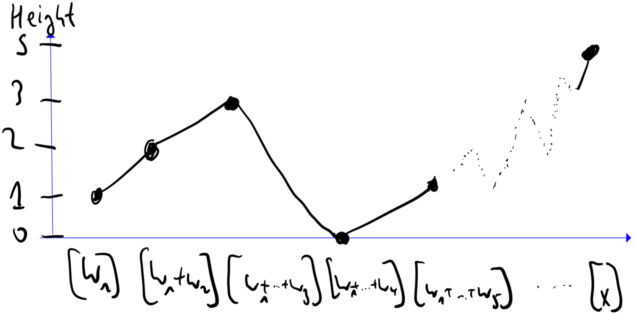
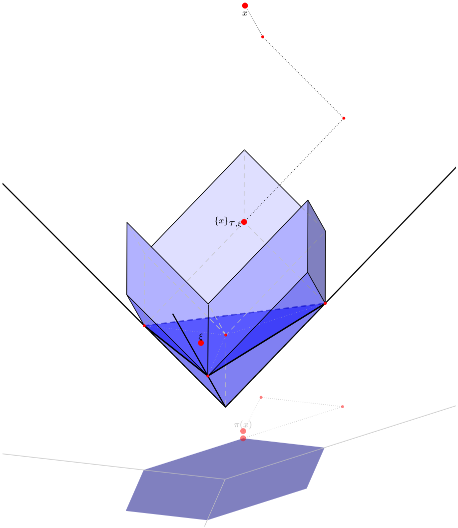


Each step goes up at most by one - all intermediate values are hit.

Method of proof: half-open triangulations

The h^*-vector counts lattice points in union of half-open parallelepipeds



Key idea of proof

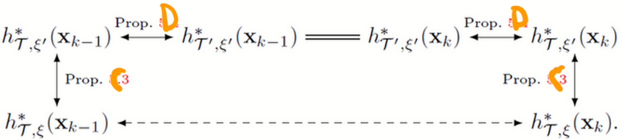


FIGURE 2. How to fill the gap between $h^*_{\mathcal{T},\xi}(\mathbf{x}_{k-1})$ and $h^*_{\mathcal{T},\xi}(\mathbf{x}_k)$ in the proof of Theorem B.

Using the circuit relation leads to step functions

$$f\colon \mathbb{R} \to \mathbb{Z}; t \mapsto \sum_{i \in I} \underline{\{} x_i - \lambda_i t \underline{\}} + \sum_{j \in J} \underline{\{} x_j + \mu_j t \underline{\}}$$

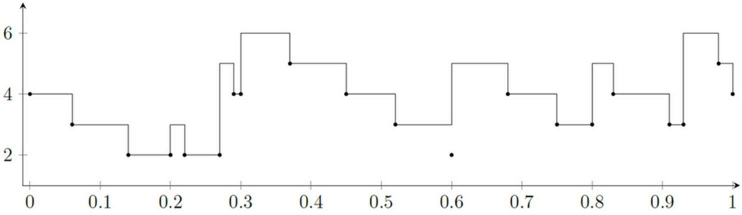


FIGURE 6. The periodic bounded step function of ex:step-fct. The dots indicate the value of f at the potential jump discontinuities.

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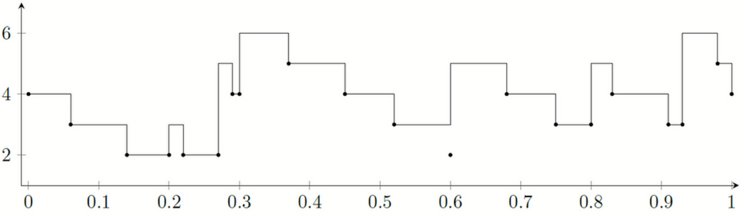

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