Ehrhart Theory for Spanning Lattice Polytopes

„Übersicht"

Preprint soon (w/ Hofscheier & Katthän)

Notation: $P \subset \mathbb{R}^d$ d-dim. lattice polytope

$$\sum_{k=0}^{\infty} |kP \cap \mathbb{Z}^d| \, t^k = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

$$h_P^* = (\underset{\substack{\| \\ 1}}{h_0^*}, \ldots, \underset{\substack{\# \\ 0}}{h_s^*}, 0, \ldots, 0), \quad s = \deg(P)$$

PAFA

Motivation

$s = 0 \iff P \cong \Delta_d$ unimodular simplex

$s = 1$ known [BN'08]

Q: $h_i^* = 0 \overset{\Longrightarrow}{\Longleftarrow}$?

$i=1$: $h_1^* = 0 \iff |P \cap \mathbb{Z}^d| = d+1$

$\iff$ P empty simplex

Def. ~~$\Lambda_P := \{$integer affine comb's of $P \cap \mathbb{Z}^d\} \subseteq \mathbb{Z}^d$~~

$\tilde{P} :=$ lattice polytope given

Sy ~~$\mathrm{conv}(P)$~~ w.r.t. Λ_P

$\rightsquigarrow$

1.

P empty ~~simplex~~ $\overset{2.}{\Longleftarrow}$ $\tilde{P}$ unim. simplex

P empty simplex $\iff$ P unimod. simplex $\iff \boxed{\deg(\tilde{P}) = 0}$

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i=2: **Prop** (Blekherman, Smith, Velasco '16) [SIAG 2013] (Journal of Squares? ...) Jans ↳ "our notation"

$h_2^* = 0 \iff \deg(\tilde{P}) \le 1$ and P is 2-IDP

$m \in 2P \cap \mathbb{Z}^d \Rightarrow$ ex $m_1, m_2 \in P \cap \mathbb{Z}^d$: $m = m_1 + m_2$

~~Example~~

~~ex. $h_P^* = (1, 0, \dots, 0, 1, 0, \dots, 0)$ and $d = 2s-1$~~

~~$d \ge 5$~~

~~not possible otherwise $h_p^* = (1, 1, 0, \dots, 1, 0, \dots, 0)$~~

i≥3: **Cor.** ∘ $h_i^* = 0 \implies \deg(\tilde{P}) \le i-1$

∘ — " — $\Leftarrow$ — " — & P i-IDP

Cannot be improved

Our result P spanning $:\iff \Lambda_P = \mathbb{Z}^d$

e.g. $\tilde{P}$ spanning

Thm P spanning $\implies h_0, \dots, h_s \neq 0$

(no gaps)

Crelle

Cor. follows from $\tilde{h}_i^* \le h_i^* \;\forall i$ (Lemma)

Conj ...

Context:

$$\begin{array}{ccc} \text{IDP} & \Longrightarrow & h^* \text{ unimodal} \\ \Downarrow & \not\nearrow & \Downarrow \\ \text{Spanning} & \Longrightarrow & h^*_0, \ldots, h^*_s \neq 0 \end{array}$$

ALL JUST IMPLICATIONS

~~Skip~~ Application

P spanning, S/I 0-algebra gen by $(P \times 1) \cap \mathbb{Z}^{d+1}$

$$\underbrace{\operatorname{reg}(S/I)}_{s} + \underbrace{\operatorname{codim}(S/I)}_{h^*_1} \overset{?}{\leq} \deg(S/I) = h^*_0 + \ldots + h^*_s$$

Eisenbud-Goto-conj (wrong)

NOT TRUE

$$\Longrightarrow \quad s + h^*_1 \leq h^*_0 + \ldots + h^*_s$$

$$\Longleftrightarrow \quad s - 1 \leq \underbrace{h^*_2}_{\geq 1} + \ldots + \underbrace{h^*_s}_{\geq 1} \quad \checkmark$$

Application to Finiteness

Needs spanning:

Ex. $h^* = (1, 0, \ldots, 0, *, 0, \ldots, 0)$

s arbitrary

MORE APPLICATIONS — NO PROOF

SIMPLEX DELETIONS VON FOLIEN

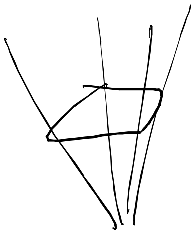
AFTER Only

h^*-quasi triangulations: h^*-vector counts lattice pts in halfopen parallelepipeds

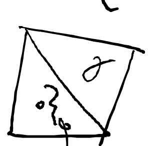
→ particular ABOVE

$\mathcal{C}$ $\quad$ $\mathcal{T}$

$d = 2$



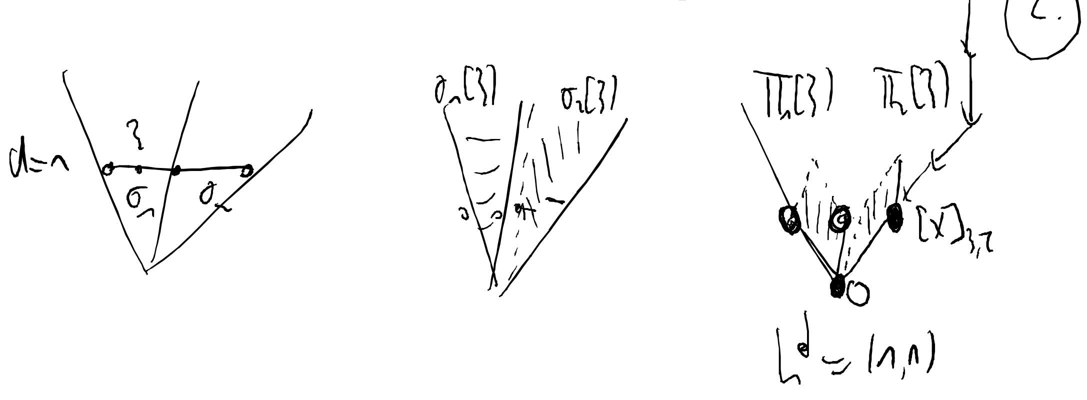
$P \times 1$



ξ generic w.r.t. $\mathcal{T}$

x_0 $\quad$ (?)

ξ generic w.r.t. τ

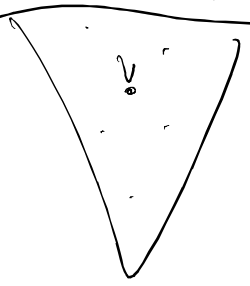


$$h^*_{\xi,\tau} : C \cap \mathbb{Z}^{d+1} \longrightarrow \mathbb{N}$$

$$X \longmapsto ht\left([X]_{\xi,\tau}\right)$$

$\boxed{\text{Thm}}$ $\forall v \in C \cap \mathbb{Z}^{d+1} \Longrightarrow$

$h^*_{\xi\tau}\left((v+\Gamma_P)\cap C\right)$ is interval.



CB $\Longrightarrow$ Thm A: $\Gamma_P = \mathbb{Z}^{d+1} \Longrightarrow$ {all possible heights} interval

$= \{ i : h^*_i \neq 0 \}$

Steps in proof:

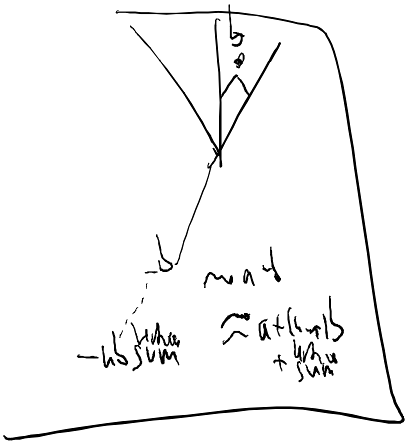
X_1 X_n X_2

Steps in proof:

X_1 X_2 X_r

similar tricks as above $[V]$, $[V+V_1]$, ..., $[V+V_1+\dots+V_r]$

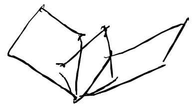
minimal   maximal



SCHEPERS UND VAN LANGENHOVEN IDEA

Problem: [illegible]

⇝ change ξ and T!

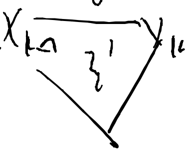


$$\begin{tikzcd}
h^*_{\mathcal{T},\xi'}(\mathbf{x}_{k-1}) \arrow[r, leftrightarrow, "\text{Prop. 5.4}"] \arrow[d, leftrightarrow, "\text{Prop. 5.3}"] & h^*_{\mathcal{T}',\xi'}(\mathbf{x}_{k-1}) \arrow[r, equal, "\text{Choose}"] & h^*_{\mathcal{T}',\xi'}(\mathbf{x}_k) \arrow[r, leftrightarrow, "\text{Prop. 5.4}"] & h^*_{\mathcal{T},\xi'}(\mathbf{x}_k) \arrow[d, leftrightarrow, "\text{Prop. 5.3}"] \\
h^*_{\mathcal{T},\xi}(\mathbf{x}_{k-1}) \arrow[rrr, leftrightarrow, dashed] & & & h^*_{\mathcal{T},\xi}(\mathbf{x}_k).
\end{tikzcd}$$

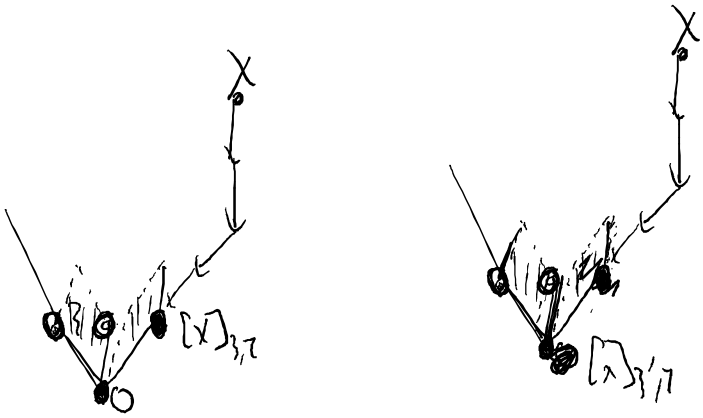
Figure 2. How to fill the gap between $h^*_{\mathcal{T},\xi}(\mathbf{x}_{k-1})$ and $h^*_{\mathcal{T},\xi}(\mathbf{x}_k)$ in the proof of Theorem 4.7. B

Choose T'

problem X_{k-1} σ' X_k ξ'



Prop C with: Changing the generic vector, gaps can be filled.



ht 0   ht 1

generic ξ: ht 0 ξ': ht 2

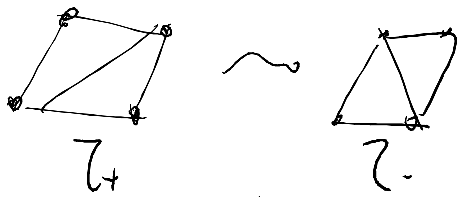
ξ ξ'

ex. ξ'': ht 1

Prop. D when changing the triang., gaps could follow

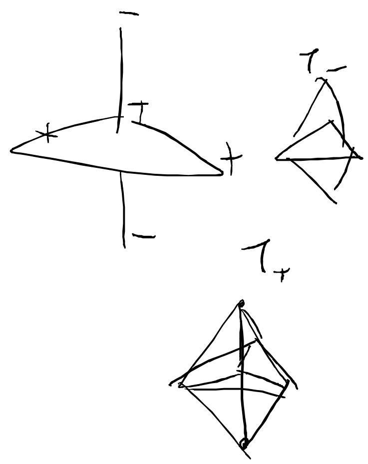
Regular triangs are connected by flips

work ~> circuit: $|P \cap \Gamma^{d}| = d+2$



Problem: $\left| h^{d}_{T_+,3}(x) - h^{d}_{T_-,3}(x) \right| \geq 2$ possible! (cannot make up by change?)

Solution: Circulation $-\sum \lambda_i v_i + \sum \mu_j v_j = 0$



	v_0	$\cdots$			v_{d+2}
0	$-\lambda_0$	$\cdots$	μ_{d+1}	$\cdots$	μ_{d+2}
X	x_0 rotation	$\cdots$ x_d	x_{d+1}		x_{d+2}

Start X | 0 | in T this cone of T_- ⟶ ht maybe 4

!

Final X | Interior cone of T_+ | 0 ⟶ ht maybe 2

$X + t_0$ | $x_0 - t\lambda_0 \cdots \cdots x_{d+2} + t\mu_{d+2}$

~>

$$f\colon \mathbb{R} \to \mathbb{Z};\, t \mapsto \sum_{i\in I} \underline{\{} x_i - \lambda_i t \underline{\}} + \sum_{j \in J} \underline{\{} x_j + \mu_j t \underline{\}},$$

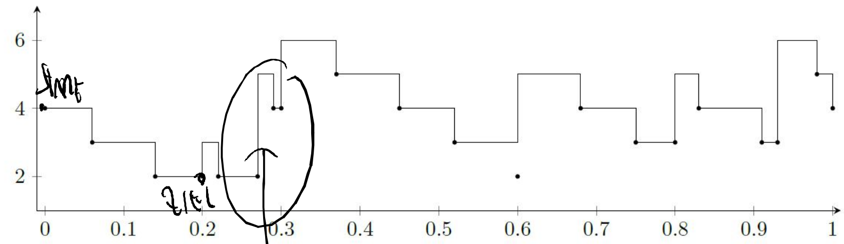


FIGURE 8. The periodic bounded step function of Example 5.16. The dots indicate the value of f at the potential jump discontinuities.

Still Problem

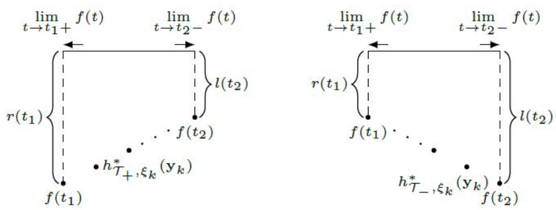


FIGURE 7. The possible cases for two successive potential jump discontinuities.

Lemma

Cons d_i := max ....

$h_i^* = 0$

$\Rightarrow$ deg $h_i \leq$ ..

$d \geq O(i)$

$\Rightarrow$ Parallel weak Cayley configuration

$h_i^* = 0$ & $d \geq O(i)$

deg $P \leq i-1$

deg $h_i(t) \leq i-1$

Cons deg := max (deg h_i) $\leq i-1$

$\Rightarrow$ Parallel weak Cayley conf.

$\Rightarrow$ Parallel Cayley conf.

Let P be ... P

wieso keine irrat. zerlegung?
Gleiches Vorgehen, wuerde nur
unwesentlich kuerzer; hat aber explizite
Beschreibung der Parallelepipede;
Lemmas unabhaengig interessant,
z.B. dadurch h-vector triangulierung