Universal Gröbner bases and Cartwright-Sturmfels ideals

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[1] “Universal Gröbner Bases for Maximal Minors”, International Mathematics Research Notices 2014

[2] “Universal Gröbner bases and Cartwright-Sturmfels ideals”, preprint 2016

[3] “Multigraded gins of determinantal ideals”, preprint 2016

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Remark: Every ideal has a Universal Gröbner basis, BUT "natural" Universal Gröbner bases are rare

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Variants: generic symmetric matrix, generic skew-symmetric matrix (and ideals of pfaffians), generic Hankel matrices
They appear in various contexts, e.g.

- classical invariant theory,
- $t = 2$: defining ideal of the Segre/Veronese/Grassmannian variety,
- higher t: secant varieties of Segre/Veronese/Grassmannian variety.

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But they are not universal GB in general!

Universal GB for minors, $t = 2$

Theorem (Sturmfels, Villareal)

Universal GB of $I_2(X)$ is the set of all the binomials associated to the cycles of the complete bipartite graph $K_{m,n}$

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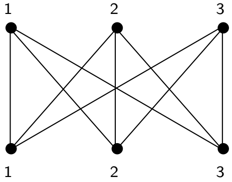
$K_{3,3}$:



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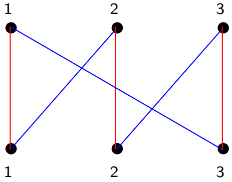

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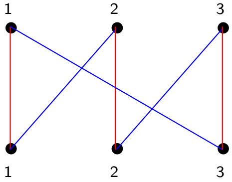

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All the initial ideals of $I_2(X)$ are radical and define CM rings (indeed they are associated to a shellable simplicial complex)

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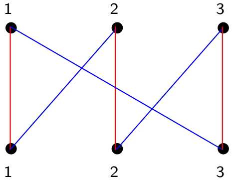
$K_{3,3}$: 

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Universal GB for maximal minors

$X = (x_{ij})$ generic matrix of size $m \times n$, $m \leq n$.

a minor of size m is called maximal minor: $[c_1, \ldots, c_m]$

$I_m(X) =$ (maximal minors of X)

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Boocher (2011)

For every term order τ:

- $\beta_{ij}(I_m(X)) = \beta_{ij}(\mathrm{in}_\tau(I_m(X)))$
- in particular $\mathrm{in}_\tau(I_m(X))$ has a linear resolution

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Fix a term order.
Consider $y_1, \dots, y_n$ are new indeterminates, and the map ϕ that sends x_{ij} to $*y_j$ where $*$ is generic scalar.

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$\phi(I_m(X)) = J$ with
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Let $D = (\mathrm{in}_\tau([c_1, \dots, c_m]) : 1 \leq c_1 < \dots c_m \leq n) \subseteq \mathrm{in}(I_m(X))$.
One has $\phi(D) = J$

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This forces equality of the Hilbert series $\implies$ $D = \mathrm{in}_\tau(I_m(X))$

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Is it possible to prove similar results for matrices of linear forms?

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Is it possible to prove similar results for matrices of linear forms?

Let $L = (L_{ij})$ an $m \times n$ matrix, $m \leq n$, with $L_{ij} \in S_1$

Eagon-Northcott

$\text{height}(I_m(L)) \leq \text{height}(I_m(X)) = n - m + 1$

If $=$ holds, then the Eagon-Northcott complex is a minimal free resolution of $I_m(L)$

What can go wrong

Example 1

$$L = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & 0 & x_4 \end{pmatrix} \qquad m = 2, n = 4, \text{height}(I_2(L)) = 2 < 3$$

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Example 2

$$L = \begin{pmatrix} x_1 + x_2 & x_3 & x_3 \\ 0 & x_1 & x_2 \end{pmatrix} \quad m = 2, n = 3, \ \text{height}(I_2(L)) = 2$$

$\text{in}_\tau(I_2(L))$ has a generator in degree 3 for every τ (if $\text{char}(K) \neq 2$)
$\implies$ $I_2(L)$ has no GB of quadrics

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$$L = \begin{pmatrix} x_1 & x_4 & x_3 \\ x_5 & x_1 + x_6 & x_2 \end{pmatrix}$$

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But $\mathrm{in}_\tau(I_2(L))$ has a generator in degree 3 for every τ with $x_1 \succ x_2 \succ \cdots \succ x_6$

$\Longrightarrow$ the 2-minors are not a UGB

Our generalizations

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Column-graded

$\deg x_{ij} = e_j \in \mathbb{Z}^n$.

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Example: $$L = \begin{pmatrix} x_{11} & 0 & x_{13} - 2x_{23} & -x_{24} \\ 0 & x_{12} + x_{22} & x_{23} & -x_{24} \end{pmatrix}$$

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Row-graded versus column-graded

Column-graded case: all the minors have distinct multidegrees

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Example

Consider $K[x_{ij} : i = 1, 2,\ j = 1, 2, 3]$ multigraded by
$\deg(x_{11}) = \deg(x_{12}) = \deg(x_{13}) = (1, 0)$
$\deg(x_{21}) = \deg(x_{22}) = \deg(x_{23}) = (0, 1)$

$$L = \begin{pmatrix} x_{11} & 2x_{11} + x_{12} & -x_{11} + x_{13} \\ x_{21} & x_{21} + x_{22} & x_{21} + x_{23} \end{pmatrix}$$

The 2 minors of L have all degree $(1, 1)$
If $x_{11} \succ x_{21} \succ \ldots$, then $\mathrm{in}_\tau(f) = x_{11}x_{21}$ for every 2-minor f

Thus the 2-minors cannot be a universal GB!

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CDG1: Results on Universal Gröbner basis for column and row-graded matrices but in the row-graded case under the assumption "maximal height"

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4. $I_2(L)$ is radical. Moreover, $\mathrm{in}_\tau(I_2(L))$ is radical for every τ
5. $I_2(L)$ has a univ. Gröbner basis of elements of multidegree $\leq \mathbf{1}$

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Properties:

- $\mathrm{gin}(I)$ is Borel fixed, that is, it is fixed by every g in $B_n(K) = \{\text{upper triangular invert. matrices}\} \subset \mathrm{GL}_n(K)$
- $\mathrm{HS}(I, y) = \mathrm{HS}(\mathrm{gin}(I), y)$

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Fix a term order (such that $x_{ij} > x_{ik}$ for every $1 \leq j < k \leq n_i$.)

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Fix a term order (such that $x_{ij} > x_{ik}$ for every $1 \leq j < k \leq n_i$.)

Theorem/Definition

$\mathrm{G} = \mathrm{GL}_{n_1}(K) \times \cdots \times \mathrm{GL}_{n_v}(K)$ acts by linear substitution on S preserving the multigraded structure.
$g \in \mathrm{G}$, $I \subset S$ multigraded ideal $\to g(I)$ (multigraded)

Multigraded generic initial ideal

$S = K[x_{ij} : i = 1, \dots, v \text{ and } j = 1 \dots, n_i]$
multigraded by $\deg(x_{ij}) = e_i \in \mathbb{Z}^v$

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As g varies in G compute $\mathrm{in}(g(I))$

For almost all g one gets the same outcome $\to \mathrm{multigin}(I)$

Borel-fixed ideals

Properties of $\mathrm{multigin}(I)$:

- $\mathrm{multigin}(I)$ is Borel fixed, that is, it is fixed by every g in

$$B_{n_1}(K) \times \cdots \times B_{n_v}(K) \subset \mathrm{G}$$

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From now on we always consider the multigraded situation.
We write gin instead of $\mathrm{multigin}$

Rigidity property of Radical Borel-fixed ideals

Theorem

Let I, J multi graded Borel-fixed ideals with $\mathrm{HS}(I, y) = \mathrm{HS}(J, y)$.

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Let J be Borel-fixed and radical. I such that $\mathrm{HS}(I, y) = \mathrm{HS}(J, y)$. Then $\mathrm{gin}(I) = J$ wrt any term order.

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(a) I is radical

(b) J has a linear resolution $\Longrightarrow$ I has a linear resolution

Cartwright-Sturmfels ideals

$S = K[x_{ij} : i = 1, \ldots, v \text{ and } j = 1 \ldots, n_i], \quad \deg(x_{ij}) = e_i \in \mathbb{Z}^v$

All the ideals are $\mathbb{Z}^v$-graded

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Ideals in CS

Examples: $X = (x_{ij})$ generic, $m \times n$.

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Why Cartwright - Sturmfels?

[Cartwright - Sturmfels, 2010]:

J such that $\mathrm{HS}(J, y) = \mathrm{HS}(I_2(X), y) \implies J$ is radical

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I square free monomial, I^* its Alexander dual

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Moreover, if $I \in \mathrm{CS}$, then $\mathrm{gin}(I)^* = \mathrm{pol}(\mathrm{gin}(I^*))$.

Ideals in CS*

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$I \in \mathrm{CS}^*$, $C = (x_{11}^{a_1} \cdots x_{v1}^{a_v} : a \in \mathbb{Z}^v \text{ and } I_a \neq 0)$

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Proof: Let τ be any term order, $H = \mathrm{in}_\tau(I)$.

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their generators have incomparable degrees $\implies$ any minimal system of generators of I is a Gröbner basis of I wrt to τ

Closure of CS and CS*

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Rmk If $I \in \mathrm{CS}^*$ and L is a $\mathbb{Z}^v$-graded linear form, then $I + (L) \notin \mathrm{CS}^*$ in general.

Example: $(x_{11}) \in \mathrm{CS}^*$, but $(x_{11}, x_{12}) \notin \mathrm{CS}^*$ because ideals in CS^* have generators with incomparable degrees.

Example 2

Rmk: F a product of $\mathbb{Z}^v$-graded linear forms:

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But if F is a $\mathbb{Z}^v$-graded form: $I : F \notin \mathrm{CS}$ in general.

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Example $S = K[x_{ij} \mid 1 \leq i, j \leq 3]$ with $\deg x_{ij} = e_i$

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix} \qquad I = I_2(X) \in \mathrm{CS}$$

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$F = x_{11}x_{21}x_{32} + x_{13}x_{23}x_{33} \to I : F = I + (x_{12}x_{13}, x_{11}x_{13})$

$\deg(x_{12}x_{13}) = (2, 0, 0) \quad \Longrightarrow \quad I : F \notin \mathrm{CS}$

Maximal minors, row-graded

Thm 1

$L = (L_{ij})$ row-graded, of size $m \times n$ with $m \leq n$. Then:

(a) There is a universal GB of elements of degree $\mathbf{1}$

(b) $I_m(L)$ is radical and it has a linear resolution. The same it is true for all its initial ideals.

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$$S \supset I_m(L) \simeq I_m(Y) + (y_{ij} - L_{ij})/(y_{ij} - L_{ij}) \subset R/(y_{ij} - L_{ij})$$

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Idea of the proof:

Let $Y = (y_{ij})$ be an $m \times n$ matrix of variables

$R = S[Y]$ $\mathbb{Z}^m$-graded by $\deg(y_{ij}) = e_i \in \mathbb{Z}^m$

Y generic $\implies$ $I_m(Y) \in \mathrm{CS}$

$$S \supset I_m(L) \simeq I_m(Y) + (y_{ij} - L_{ij})/(y_{ij} - L_{ij}) \subset R/(y_{ij} - L_{ij})$$

$$I_m(Y) \in \mathrm{CS} \implies I_m(L) \in \mathrm{CS} \implies \mathrm{in}_\tau(I_m(L)) \in \mathrm{CS} \text{ for all } \tau$$

Why column-graded is easier that row-graded?

Thm 1

Assume $L = (L_{ij})$ is column-graded of size $m \times n$ with $m \leq n$. Then:

(a) the maximal minors of L form a universal GB

(b) $I_m(L)$ is radical and it has a linear resolution. The same it is true for all its initial ideals.

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Moreover, in the column-graded case:
if $Y = (y_{ij})$ is generic, $m \times n$ with $m \leq n$, then $I_m(Y) \in \mathrm{CS}^*$.

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It follows that $I_m(L) \in \mathrm{CS} \cap \mathrm{CS}^*$.

Equality of Betty numbers

Recall: $I \in \mathsf{CS}^* \implies \beta_{i,\mathbf{a}}(I) = \beta_{i,\mathbf{a}}(\mathrm{in}_\tau(I))$ for every τ.

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Not true for 2-minors in general

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[CDG1]:
column-graded:

$$\mathrm{gin}(I_m(L)) = (x_{1j_1} \cdots x_{mj_m} \mid [j_1, \ldots, j_m]_L \neq 0)$$

row-graded, height$(I_m(L)) = n - m + 1$:

$$\mathrm{gin}(I_m(L)) = (x_{1j_1} \cdots x_{mj_m} : j_1 + \cdots + j_m \leq n)$$

Generic initial ideals [CDG3]

Maximal minors, row-graded

$L = (L_{ij})$ row-graded $m \times n$, $m \leq n$, $I = I_m(L)$

$A \subseteq [m]$:

$b(A) = \dim_K <$ columns of the matrix $L_A = (L_{ij})_{i \in A, j \in [n]} >$

$$\mathrm{gin}(I) = (x_{1b_1} \cdots x_{mb_m} : \sum_{i \in A} b_i \leq b(A) \text{ for every } A \subseteq [m])$$

Generic initial ideals [CDG3]

2-minors

$L = (L_{ij})$ row or column-graded, $I = I_2(L)$

$$\mathrm{gin}(I) = (\prod_{i \in A} x_{ib_i} : A \subseteq [m] \text{ and } b_i \text{ satisfying } *)$$

$$* : \begin{cases} 1 \leq b_i \leq n - \dim_K V_i \text{ for every } i \in A \\ \sum_{i \in A} b_i \leq n(|A| - 1) + \dim_K V_A - \sum_{i \in A} \dim_K V_i \end{cases}$$

with $V_i = \{\lambda \in K^n | \sum_{i=1}^{n} \lambda_j L_{ij} = 0\}$ and $V_A = \sum_{i \in A} V_i$.