Some recent interactions between commutative algebra and combinatorics

FABRIZIO ZANELLO
(Michigan Tech)

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Graded algebras

- Let $A = R/I = \bigoplus_{i \geq 0} A_i$ be a standard graded k-algebra, i.e., $R = k[x_1, \ldots, x_r]$, $A_0 = k$ is a field, $I \subset R$ is a homogeneous ring ideal, and $\deg(x_i) = 1$.
- Often char(k) has suitable restrictions (e.g., $= 0$).
- The Hilbert function (HF) of A is $H_A(i) = \dim_k A_i$.
- The Hilbert series of A is $\sum_{i=0}^{\infty} H_A(i) t^i$, i.e., it's the generating function of the HF of A.
- It's a standard fact from commutative algebra that the Hilbert series of A is a rational function with numerator $\sum_{i=0}^{e} h_i t^i$, where the h_i are integers and $h_e \neq 0$.
- The sequence $h_A = (h_0 = 1, h_1, \ldots, h_e)$ is called the h-vector of A, and the index e is its socle degree.

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Artinian algebras

- A is artinian if it's Cohen-Macaulay of Krull-dimension zero (i.e., every homogeneous prime ideal of A is maximal), or $\sqrt{I} = (x_1, \ldots, x_r)$, or $H_A(i) = 0$ for $i \gg 0$ (all equivalent; we'll mostly use the latter).
- Thus, in the artinian case, the h-vector and HF of A coincide. In particular, the h_i are nonnegative.
- The h-vector of an artinian algebra is also called an O-sequence (M-sequence, sometimes).
- A classical theorem of Macaulay (Proc. London Math. Soc. 1927) characterized all possible O-sequences.

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Level and Gorenstein algebras

- The socle of A artinian is $\mathrm{soc}(A) = 0 : (\overline{x_1}, \ldots, \overline{x_r}) \subset A$.
- The socle-vector of A is the HF of the socle; that is, $s(A) = (0, s_1, \ldots, s_e)$, where $s_i = \dim_k \mathrm{soc}(A)_i$.
- A is level (of type t) if $s(A) = (0, \ldots, 0, t)$.
- A is Gorenstein if it's level of type 1; i.e., when $s(A) = (0, \ldots, 0, t = 1)$.
- Equivalent homological definitions: A is level when the last module of the graded minimal free resolution of A contains only one shift; it's Gorenstein when that unique shift appears with multiplicity one.
  We won't go into this today, but see e.g. the monograph of Geramita *et al.* (AMS Memoir 2007) for one of the first works developing an homological approach to the study of level HFs.

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Example

- **Example 1.**

$$A = k[x,y]/(x^3, xy^2, y^4)$$

has HF $h = (1,2,3,2)$ and socle-vector $s = (0,0,0,2)$. Hence A is (monomial) level of type 2.

- **Example 2.**

$$A = \mathbb{C}[x,y,z]/(z^3, 2x^2z + 3yz^2, 2x^2y + 3y^2z, y^3, x^4)$$

has HF $h = (1,3,6,6,3,1)$ and socle-vector $s = (0,0,0,0,0,1)$. Hence A is Gorenstein.

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The general theme of the talk...

The main (very) general question of the talk is:
Which O-sequences are the Hilbert functions of *level* (or *Gorenstein*) algebras? How about in the *monomial* case?

- The study of level algebras and their HFs is important not only in its own right, but because of many applications to other fields, including algebraic and enumerative combinatorics, plane partitions, matroid theory, design theory and finite geometries, algebraic geometry, etc..
- Some of these applications have just recently been discovered or studied systematically. We'll try to discuss a few of the more exciting combinatorial developments in this talk.

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Unimodality

- A finite sequence (of nonnegative integers) $a_0, a_1, \ldots, a_n$ is unimodal if

$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n,$$

for some index k, $0 \leq k \leq n$.

- Unimodality is a central concept in combinatorics, algebra and other fields. Many important results on level algebras and subsets thereof concern unimodality issues. For a wealth of info, examples and techniques, see the two classics: "Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry", by R. Stanley (1989, written in 1986); and "Same title: an update", by F. Brenti (1994).

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Arbitrary level and Gorenstein HFs

- In codimension $r = 2$, level and Gorenstein HFs are completely characterized. (In fact, so are the HFs associated to any given socle-vector; see e.g. A. Iarrobino, TAMS 1984.)
- In codimension $r = 3$, trouble starts. Essentially the only nice characterization is still that of Gorenstein HFs: They are exactly the SI-sequences, i.e., they are symmetric and their first half is differentiable. (A sequence is *differentiable* if its first difference is an O-sequence.) See Stanley, Adv. Math. 1978, and then FZ, PAMS 2006 for a combinatorial proof.
- In codimension 3, Geramita *et al.* (AMS Memoir 2007) characterized all level HFs of socle degree 5, and those of socle degree 6 and type 2, by a variety of techniques.

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Nonunimodal examples

- For for any $r \geq 3$, level HFs can be nonunimodal in the "worst possible way", i.e., with arbitrarily many peaks (FZ, J. Algebra 2006). In particular, a classification of such HFs seems entirely out of reach.
- The above result answered a question from the Geramita *et al.* 2007 Memoir, and was later improved in several directions, both combinatorial and geometric (see e.g. Migliore, Canad. J. Math. 2008).
- The state of the art on nonunimodal level HFs for $r = 3$ and $r = 4$ is A. Weiss' PhD thesis (Tufts, 2007).
- Weiss constructed classes of nonunimodal HFs of any type $t \geq 5$ for $r = 3$, and of any type $t \geq 3$ for $r = 4$, by nicely extending some *inverse system* ideas of Iarrobino and myself.

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Codimension 3 ad 4

- Unimodality in codimension $r = 3$ is open for types 2, 3 and 4. For $r = 4$, unimodality is open for types 1 and 2.
- In particular, proving the unimodality of Gorenstein HFs of codimension 4 is an important open problem in this area. See Iarrobino-Srinivasan (JPAA 2005), Migliore-Nagel-FZ (MRL 2008) and Seo-Srinivasan (Comm. Algebra 2012) for some progress, mostly in characteristic zero.
- Another important open question: Are all level HFs of codimension 3 flawless? (Recall that $h = (1, h_1, \dots, h_e)$ is *flawless* if $h_i \leq h_{e-i}$ for all $i \leq e/2$.)
  If the answer is "no", this will have striking algebraic consequences on the Lefschetz properties for Gorenstein algebras (BMMNZ2, J. Algebra 2014).

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Higher codimension

- For $r \geq 5$, very little is known. Stanley (Adv. Math. 1978), Bernstein-Iarrobino (Comm. Algebra 1992), Boij-Laksov (PAMS 1994), etc., proved the existence of nonunimodal level and Gorenstein HFs for all types $t \geq 1$ and codimensions $r \geq 5$. In particular, these HFs are extremely hard to characterize.
- For instance, given r, in general we don't even know the minimum $f(r)$ for a Gorenstein HF $(1, r, f(r), r, 1)$.
- Migliore and I recently proved (arXiv:1512.01433; PAMS, to appear) that Stanley's 1978 nonunimodal example, $(1, 13, 12, 13, 1)$, is the smallest possible.
- Asymptotically: $f(r) \sim_{r \longrightarrow \infty} (6r)^{2/3}$, as conjectured by Stanley, 1983 (see Migliore-Nagel-FZ, PAMS 2008).
- However, regardless of $f(r)$, $(1, r, c, r, 1)$ is Gorenstein if and only if $c \in \left[f(r), \binom{r+1}{2} \right]$ (FZ, J. Algebra 2009).

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The Interval Property

- Interval Conjecture (IC) (FZ, J. Algebra 2009).
  Suppose that for some $\alpha \geq 0$, $(1, h_1, \ldots, h_i, \ldots, h_e)$ and $(1, h_1, \ldots, h_i + \alpha, \ldots, h_e)$ are both level HFs.
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How much does the characteristic matter?

- The IC and GIC seem consistent with all existing techniques for level and Gorenstein HFs, and would finally provide a strong structural result on their sets.
- The Interval Property is true in many instances; however, it's false in general for pure O-sequences (conjectured in BMMNZ, and disproved by Varbaro-Constantinescu), but true (and consequential) in socle degree 3; false also for pure f-vectors (Pastine-FZ, PAMS 2015); false in general for r-differential posets (Stanley-FZ, E-JC 2012), but open for 1-differential posets, which is the main class.
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Pure O-sequences

- A monomial order ideal is a nonempty, finite set X of (monic) monomials, such that for $M \in X$ and N dividing M, we have $N \in X$. (It's a special case of an order ideal in a poset.)
- The h-vector of X is its degree vector, $h = (1, h_1, \ldots, h_e)$, counting the monomials of X in each degree.
- A pure O-sequence is the h-vector of a monomial order ideal X whose maximal (by divisibility) monomials all have the same degree.
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Example

- The pure monomial order ideal (inside $k[x, y, z]$) with maximal monomials xy^3z and x^2z^3 is:

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; \quad y^3z, xy^2z, xy^3, xz^3, x^2z^2;$$

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Some old and new results

- In BMMNZ, AMS Memoir 2012, we attempted to develop a systematic theory of pure O-sequences. Very few results were known before, and many are still unknown now. Among the main previous results:
- THEOREM (Hibi). Let $h = (1, h_1, \ldots, h_e)$ be a pure O-sequence. Then $h_i \leq h_j$ for all $0 \leq i \leq j \leq e - i$.
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- *NOTE 1. Differentiability holds characteristic free, though Hausel's proof was of a "g-theorem" in characteristic zero.*
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Pure f-vectors

- A collection Δ of subsets of $V = \{v_1, \ldots, v_n\}$ is a simplicial complex if, for each $F \in \Delta$ and $G \subseteq F$, we have $G \in \Delta$. (Yet another order ideal in a poset!)
- The elements of Δ are dubbed faces, and the maximal faces (under inclusion) are the facets of Δ.
- The f-vector of Δ is the vector $f(\Delta) = (1, f_0, \ldots, f_{d-1})$, where f_i is the number of cardinality $i + 1$ faces of Δ.
- A *pure* f-vector is the f-vector of a *pure* simplicial complex, i.e., one with all facets of the same cardinality.
- Equivalently, a pure f-vector is a *squarefree* pure O-sequence, hence an even smaller class of level HFs!
- Little is known about pure f-vectors. See Colbourn-Keranen-Kreher, Discrete Math. 2014, for a characterization in socle degree 3. Virtually any result might be interesting. Numerous applications, including to finite geometries and combinatorial designs...

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Matroid simplicial complexes

- Let k be a field and Δ any simplicial complex over $V = \{v_1, \dots, v_n\}$. We can associate to Δ a squarefree monomial ideal in $S = k[x_1, \dots, x_n]$, namely

$$I_\Delta = \left(\prod_{v_i \in F} x_i \mid F \notin \Delta \right) \subseteq S.$$

- I_Δ is the Stanley-Reisner ideal of Δ, and the quotient algebra $k[\Delta] = S/I_\Delta$ is its Stanley-Reisner ring.
- Δ is a matroid complex if, for every subset $W \subseteq V$, the *restriction* $\Delta|_W$ is a *pure simplicial complex*.

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The big, unproved connection

- The h-vector of a simplicial complex Δ is defined as the h-vector of its Stanley-Reisner ring $k[\Delta]$.
- It's a standard combinatorial algebra fact that the h-vector of a matroid is level (though in general Δ has positive Krull-dimension). In fact, more is conjectured:
- CONJECTURE (Stanley 1977). A matroid h-vector is a pure O-sequence!
- A huge amount of work over the past 40 years, though only solutions of special cases. With Hà and Stokes (Annals Comb. 2013), I introduced a first possible general approach, using the theory of pure O-sequences. However, in the paper we completely solved only the Krull-dimension 2 case.
- See also, among the many interesting recent works: Constantinescu-Varbaro, De Loera-Kemper-Klee, Merino-Noble-Ramírez-Villarroel, Oh, Schweig.

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