Powers of sums of ideals

Huy Tài Hà
Tulane University

Partially supported by: Japan Society for the Promotion of Science and Bernick Faculty Grant of Tulane University

(Joint with Hop D. Nguyen, Ngo Viet Trung, Tran Nam Trung)

Problems

- Let k be a field. Let $A = k[x_1, \ldots, x_r]$ and $B = k[y_1, \ldots, y_s]$ be polynomial rings over k.
- Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper homogeneous ideals.

Problem

Investigate algebraic invariants and properties of

$$(I + J)^n \text{ and } (I + J)^{(n)} \subseteq R = A \otimes_k B$$

via invariants and properties of powers of I and J.

Motivation

- **Powers of ideals** appear naturally in singularities and multiplicity theories.
- **Fiber product:** Let $X = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B/J$. Then
$$X \times_k Y = \operatorname{Spec} R/(I + J).$$
- **Disjoint union:** Let G_1 and G_2 be simple graphs on vertex sets $V = \{x_1, \dots, x_r\}$ and $W = \{y_1, \dots, y_s\}$, and let $G = G_1 \sqcup G_2$ be their disjoint union. Then
$$I(G) = I(G_1) + I(G_2).$$
- **Hyperplane section:** $J = (y) \subseteq k[y] = B$. In this case,
$$I + J = (I, y) \subseteq k[x_1, \dots, x_r, y].$$

Motivation

- **Powers of ideals** appear naturally in singularities and multiplicity theories.
- **Fiber product:** Let $X = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B/J$. Then
$$X \times_k Y = \operatorname{Spec} R/(I + J).$$
- **Disjoint union:** Let G_1 and G_2 be simple graphs on vertex sets $V = \{x_1, \dots, x_r\}$ and $W = \{y_1, \dots, y_s\}$, and let $G = G_1 \sqcup G_2$ be their disjoint union. Then
$$I(G) = I(G_1) + I(G_2).$$
- **Hyperplane section:** $J = (y) \subseteq k[y] = B$. In this case,
$$I + J = (I, y) \subseteq k[x_1, \dots, x_r, y].$$

Motivation

- **Powers of ideals** appear naturally in singularities and multiplicity theories.
- **Fiber product:** Let $X = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B/J$. Then
$$X \times_k Y = \operatorname{Spec} R/(I + J).$$
- **Disjoint union:** Let G_1 and G_2 be simple graphs on vertex sets $V = \{x_1, \ldots, x_r\}$ and $W = \{y_1, \ldots, y_s\}$, and let $G = G_1 \sqcup G_2$ be their disjoint union. Then
$$I(G) = I(G_1) + I(G_2).$$
- **Hyperplane section:** $J = (y) \subseteq k[y] = B$. In this case,
$$I + J = (I, y) \subseteq k[x_1, \ldots, x_r, y].$$

Motivation

- **Powers of ideals** appear naturally in singularities and multiplicity theories.
- **Fiber product:** Let $X = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B/J$. Then
$$X \times_k Y = \operatorname{Spec} R/(I + J).$$
- **Disjoint union:** Let G_1 and G_2 be simple graphs on vertex sets $V = \{x_1, \ldots, x_r\}$ and $W = \{y_1, \ldots, y_s\}$, and let $G = G_1 \sqcup G_2$ be their disjoint union. Then
$$I(G) = I(G_1) + I(G_2).$$
- **Hyperplane section:** $J = (y) \subseteq k[y] = B$. In this case,
$$I + J = (I, y) \subseteq k[x_1, \ldots, x_r, y].$$

Motivation

Problem

Suppose that

- $I \subseteq A = k[x_1, \ldots, x_r]$ is a homogeneous ideal,
- y is a new variable, and
- $R = A[y]$.

Compute the depth and regularity of $(I, y)^n$ and $(I, y)^{(n)}$ in R.

Algebraic invariants

Definition

Let R be a standard graded k-algebra, and let $\mathfrak{m}$ be its maximal homogenous ideal. Let M be a finitely generated graded R-module. Then

- $\operatorname{depth} M := \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$;
- $\operatorname{reg} M := \max\{t \mid H^i_{\mathfrak{m}}(M)_{t-i} = 0 \ \forall\, i \geq 0\}$.

- Over polynomial rings, these invariants are related to the minimal free resolution.

Algebraic invariants

Definition

Let R be a standard graded k-algebra, and let $\mathfrak{m}$ be its maximal homogenous ideal. Let M be a finitely generated graded R-module. Then

- $\operatorname{depth} M := \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$;
- $\operatorname{reg} M := \max\{t \mid H^i_{\mathfrak{m}}(M)_{t-i} = 0 \ \forall\ i \geq 0\}$.

- Over polynomial rings, these invariants are related to the minimal free resolution.

Powers of sums of ideals by approximation

- $A = k[x_1, \ldots, x_r], B = k[y_1, \ldots, y_s]$ and $R = A \otimes_k B$.
- $I \subseteq A$ and $J \subseteq B$ are proper homogeneous ideals.

Key observation: $(I + J)^n = \sum_{t=0}^{n} I^{n-t} J^t$.

- Set $Q_p := \sum_{t=0}^{p} I^{n-t} J^t$. Then

$$I^n = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I + J)^n.$$

- $Q_p / Q_{p-1} = I^{n-p} J^p / I^{n-p+1} J^p$.
- There are 2 short exact sequences

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{n-p+1}J^p \longrightarrow R/I^{n-p}J^p \longrightarrow 0.$$

Powers of sums of ideals by approximation

- $A = k[x_1, \ldots, x_r], B = k[y_1, \ldots, y_s]$ and $R = A \otimes_k B$.
- $I \subseteq A$ and $J \subseteq B$ are proper homogeneous ideals.

Key observation: $(I + J)^n = \sum_{t=0}^{n} I^{n-t} J^t$.

- Set $Q_p := \sum_{t=0}^{p} I^{n-t} J^t$. Then

$$I^n = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I + J)^n.$$

- $Q_p / Q_{p-1} = I^{n-p} J^p / I^{n-p+1} J^p$.
- There are 2 short exact sequences

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{n-p+1}J^p \longrightarrow R/I^{n-p}J^p \longrightarrow 0.$$

Powers of sums of ideals by approximation

- $A = k[x_1, \ldots, x_r], B = k[y_1, \ldots, y_s]$ and $R = A \otimes_k B$.
- $I \subseteq A$ and $J \subseteq B$ are proper homogeneous ideals.

Key observation: $(I + J)^n = \sum_{t=0}^{n} I^{n-t} J^t$.

- Set $Q_p := \sum_{t=0}^{p} I^{n-t} J^t$. Then

$$I^n = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I + J)^n.$$

- $Q_p / Q_{p-1} = I^{n-p} J^p / I^{n-p+1} J^p$.
- There are 2 short exact sequences

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{n-p+1}J^p \longrightarrow R/I^{n-p}J^p \longrightarrow 0.$$

Powers of sums of ideals by approximation

- $A = k[x_1, \ldots, x_r], B = k[y_1, \ldots, y_s]$ and $R = A \otimes_k B$.
- $I \subseteq A$ and $J \subseteq B$ are proper homogeneous ideals.

Key observation: $(I + J)^n = \sum_{t=0}^{n} I^{n-t} J^t$.

- Set $Q_p := \sum_{t=0}^{p} I^{n-t} J^t$. Then

$$I^n = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I + J)^n.$$

- $Q_p / Q_{p-1} = I^{n-p} J^p / I^{n-p+1} J^p$.
- There are 2 short exact sequences

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{n-p+1}J^p \longrightarrow R/I^{n-p}J^p \longrightarrow 0.$$

Powers of sums of ideals by approximation

- $A = k[x_1, \ldots, x_r], B = k[y_1, \ldots, y_s]$ and $R = A \otimes_k B$.
- $I \subseteq A$ and $J \subseteq B$ are proper homogeneous ideals.

Key observation: $(I + J)^n = \sum_{t=0}^{n} I^{n-t} J^t$.

- Set $Q_p := \sum_{t=0}^{p} I^{n-t} J^t$. Then

$$I^n = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I + J)^n.$$

- $Q_p / Q_{p-1} = I^{n-p} J^p / I^{n-p+1} J^p$.
- There are 2 short exact sequences

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{n-p+1}J^p \longrightarrow R/I^{n-p}J^p \longrightarrow 0.$$

Powers of sums of ideals by approximation

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{n-p+1}J^p \longrightarrow R/I^{n-p}J^p \longrightarrow 0.$$

Lemma (Hoa - Tâm)

1. $\operatorname{reg} R/IJ = \operatorname{reg} A/I + \operatorname{reg} B/J + 1.$
2. $\operatorname{depth} R/IJ = \operatorname{depth} A/I + \operatorname{depth} B/J + 1.$

Theorem

*(**—, N.V. Trung and T.N. Trung**) For all $n \geq 1$, we have*

1. $\operatorname{depth} R/(I+J)^n \geq$

$$\min_{i\in[1,n-1],\ j\in[1,n]} \{\operatorname{depth} A/I^{n-i} + \operatorname{depth} B/J^i + 1,$$

$$\operatorname{depth} A/I^{n-j+1} + \operatorname{depth} B/J^j\},$$

2. $\operatorname{reg} R/(I+J)^n \leq$

$$\max_{i\in[1,n-1],\ j\in[1,n]} \{\operatorname{reg} A/I^{n-i} + \operatorname{reg} B/J^i + 1,$$

$$\operatorname{reg} A/I^{n-j+1} + \operatorname{reg} B/J^j\}.$$

*(**Hop D. Nguyen**) If, in addition, either char $k = 0$ or I and J are monomial ideals then we have the equalities.*

Depth and Castelnuovo-Mumford regularity

Proposition

Assume that depth $A/I^2 \geq$ depth $A/I + 1$.

1. If depth $B/J^2 \geq$ depth $B/J + 1$ then depth $R/(I+J)^2 =$ depth $A/I +$ depth $B/J + 1$.
2. If depth $B/J^2 <$ depth B/J then depth $R/(I+J)^2 =$ depth $A/I +$ depth B/J^2.

Depth and Castelnuovo-Mumford regularity

Proposition

Assume that $\operatorname{reg} A/I^2 \le \operatorname{reg} A/I + 1$.

1. If $\operatorname{reg} B/J^2 \le \operatorname{reg} B/J + 1$ then $\operatorname{reg} R/(I+J)^2 = \operatorname{reg} A/I + \operatorname{reg} B/J + 1$.
2. If $\operatorname{reg} B/J^2 > \operatorname{reg} B/J$ then $\operatorname{reg} R/(I+J)^2 = \operatorname{reg} A/I + \operatorname{reg} B/J^2$.

Example (Conca)

Let $A = k[x_1, x_2, x_3]$ and $I = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4, x_1^2x_2^2x_3^5)$. Using Macaulay2, we get $\operatorname{reg} A/I = 8$ and $\operatorname{reg} A/I^2 = 7$.

Proposition

Assume that $\operatorname{reg} A/I^2 \leq \operatorname{reg} A/I + 1$.

1. If $\operatorname{reg} B/J^2 \leq \operatorname{reg} B/J + 1$ then $\operatorname{reg} R/(I+J)^2 = \operatorname{reg} A/I + \operatorname{reg} B/J + 1$.
2. If $\operatorname{reg} B/J^2 > \operatorname{reg} B/J$ then $\operatorname{reg} R/(I+J)^2 = \operatorname{reg} A/I + \operatorname{reg} B/J^2$.

Example (Conca)

Let $A = k[x_1, x_2, x_3]$ and $I = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4, x_1^2x_2^2x_3^5)$. Using Macaulay2, we get $\operatorname{reg} A/I = 8$ and $\operatorname{reg} A/I^2 = 7$.

Depth and Castelnuovo-Mumford regularity

Corollary

Assume that J is generated by variables. Then

1. $\operatorname{depth} R/(I+J)^n = \min_{i \le n} \operatorname{depth} A/I^i + \dim B/J$, *and*
2. $\operatorname{reg} R/(I+J)^n = \max_{i \le n}\{\operatorname{reg} A/I^i - i\} + n.$

Proposition

$$(I+J)^n/(I+J)^{n+1} = \bigoplus_{i+j=n} \left(I^i/I^{i+1} \otimes_k J^j/J^{j+1}\right).$$

Lemma (Goto - Watanabe)

Let M and N be graded modules over A and B, respectively. Let $\mathfrak{u}$ and $\mathfrak{v}$ be the maximal homogeneous ideals of A and B, respectively, and let $\mathfrak{m}$ be the maximal homogeneous ideal of R. Then

$$H^n_{\mathfrak{m}}(M \otimes_k N) = \bigoplus_{i+j=n} H^i_{\mathfrak{u}}(M) \otimes_k H^j_{\mathfrak{v}}(N).$$

In particular,

1. $\operatorname{depth} M \otimes_k N = \operatorname{depth} M + \operatorname{depth} N$, *and*
2. $\operatorname{reg} M \otimes_k N = \operatorname{reg} M + \operatorname{reg} N$.

Powers of sums of ideals by decomposition

Proposition

$$(I+J)^n/(I+J)^{n+1} = \bigoplus_{i+j=n} \left(I^i/I^{i+1} \otimes_k J^j/J^{j+1}\right).$$

Lemma (Goto - Watanabe)

Let M and N be graded modules over A and B, respectively. Let $\mathfrak{u}$ and $\mathfrak{v}$ be the maximal homogeneous ideals of A and B, respectively, and let $\mathfrak{m}$ be the maximal homogeneous ideal of R. Then

$$H^n_{\mathfrak{m}}(M \otimes_k N) = \bigoplus_{i+j=n} H^i_{\mathfrak{u}}(M) \otimes_k H^j_{\mathfrak{v}}(N).$$

In particular,

1. depth $M \otimes_k N$ = depth M + depth N, *and*
2. reg $M \otimes_k N$ = reg M + reg N.

Powers of sums of ideals by decomposition

Proposition

$$(I+J)^n/(I+J)^{n+1} = \bigoplus_{i+j=n} (I^i/I^{i+1} \otimes_k J^j/J^{j+1}).$$

Lemma (Goto - Watanabe)

Let M and N be graded modules over A and B, respectively. Let $\mathfrak{u}$ and $\mathfrak{v}$ be the maximal homogeneous ideals of A and B, respectively, and let $\mathfrak{m}$ be the maximal homogeneous ideal of R. Then

$$H^n_{\mathfrak{m}}(M \otimes_k N) = \bigoplus_{i+j=n} H^i_{\mathfrak{u}}(M) \otimes_k H^j_{\mathfrak{v}}(N).$$

In particular,

1. $\operatorname{depth} M \otimes_k N = \operatorname{depth} M + \operatorname{depth} N$, *and*
2. $\operatorname{reg} M \otimes_k N = \operatorname{reg} M + \operatorname{reg} N$.

Depth and Castelnuovo-Mumford regularity

Theorem (—, N.V. Trung and T.N. Trung)

For all $n \geq 1$, we have

1. $$\operatorname{depth} \frac{(I+J)^n}{(I+J)^{n+1}} = \min_{i+j=n} \{\operatorname{depth} I^i/I^{i+1} + \operatorname{depth} J^j/J^{j+1}\},$$
2. $$\operatorname{reg}(I+J)^n/(I+J)^{n+1} = \max_{i+j=n} \{\operatorname{reg} I^i/I^{i+1} + \operatorname{reg} J^j/J^{j+1}\}.$$

Corollary

The following are equivalent:

1. $R/(I+J)^t$ is Cohen-Macaulay for all $t \leq n$;
2. $(I+J)^{n-1}/(I+J)^n$ is Cohen-Macaulay;
3. A/I^t and B/J^t are Cohen-Macaulay for all $t \leq n$;
4. I^t/I^{t+1} and J^t/J^{t+1} are Cohen-Macaulay for all $t \leq n-1$.

Behaviors of depth function

Definition

An ideal is said to have a *constant depth function* if the depth of all its powers are the same.

Proposition (—, N.V. Trung and T.N. Trung)

Let I and J be squarefree monomial ideals. Then $I + J$ has a constant depth function if and only if so do I and J.

- **Herzog-Vladiou:** proved this result under an additional condition that the Rees algebras and I and J are Cohen-Macaulay.

Behaviors of depth function

Definition

An ideal is said to have a *constant depth function* if the depth of all its powers are the same.

Proposition (—, N.V. Trung and T.N. Trung)

Let I and J be squarefree monomial ideals. Then $I + J$ has a constant depth function if and only if so do I and J.

- **Herzog-Vladiou:** proved this result under an additional condition that the Rees algebras and I and J are Cohen-Macaulay.

Definition

An ideal is said to have a *constant depth function* if the depth of all its powers are the same.

Proposition (—, N.V. Trung and T.N. Trung)

Let I and J be squarefree monomial ideals. Then $I + J$ has a constant depth function if and only if so do I and J.

- **Herzog-Vladiou:** proved this result under an additional condition that the Rees algebras and I and J are Cohen-Macaulay.

Asymptotic depth and regularity

- **Brodmann:** depth $A/I^n =$ constant for $n \gg 0$.

Theorem (—, N.V. Trung and T.N. Trung)

$$\lim_{n\to\infty} \operatorname{depth} R/(I+J)^n = \min\Big\{ \lim_{i\to\infty} \operatorname{depth} A/I^i + \min_{j\geq 0} \operatorname{depth} B/J^j,\ \min_{i\geq 0} \operatorname{depth} A/I^i + \lim_{j\to\infty} \operatorname{depth} B/J^j \Big\}.$$

- **Brodmann:** depth $A/I^n =$ constant for $n \gg 0$.

Theorem (—, N.V. Trung and T.N. Trung)

$$\lim_{n\to\infty} \operatorname{depth} R/(I+J)^n = \min\Big\{ \lim_{i\to\infty} \operatorname{depth} A/I^i + \min_{j\geq 0} \operatorname{depth} B/J^j, \\ \min_{i\geq 0} \operatorname{depth} A/I^i + \lim_{j\to\infty} \operatorname{depth} B/J^j \Big\}.$$

Asymptotic depth and regularity

- **Cutkosky-Herzog-Trung; Kodiyalam; Trung-Wang:**

$$\operatorname{reg} I^n = an + b \text{ for } n \gg 0.$$

Theorem (—, N.V. Trung and T.N. Trung)

Assume that $\operatorname{reg} I^n = dn + e$ *for* $n \geq \operatorname{lin}(I)$ *and* $\operatorname{reg} J^n = cn + f$ *for* $n \geq \operatorname{lin}(J)$*, where* $c \geq d$*. Set* $e^* = \max_{i \leq \operatorname{lin}(I)}\{\operatorname{reg} I^i - ci\}$ *and* $f^* = \max_{j \leq \operatorname{lin}(J)}\{\operatorname{reg} J^j - dj\}$*. Then, for* $n \gg 0$*, we have*

$$\operatorname{reg}(I+J)^n = \begin{cases} c(n+1) + f + e^* - 1 & \text{if } c > d, \\ d(n+1) + \max\{e^* + f, e + f^*\} - 1 & \text{if } c = d. \end{cases}$$

- **Cutkosky-Herzog-Trung; Kodiyalam; Trung-Wang:**

$$\operatorname{reg} I^n = an + b \text{ for } n \gg 0.$$

Theorem (—, N.V. Trung and T.N. Trung)

Assume that $\operatorname{reg} I^n = dn + e$ *for* $n \geq \operatorname{lin}(I)$ *and* $\operatorname{reg} J^n = cn + f$ *for* $n \geq \operatorname{lin}(J)$*, where* $c \geq d$*. Set* $e^* = \max_{i \leq \operatorname{lin}(I)}\{\operatorname{reg} I^i - ci\}$ *and* $f^* = \max_{j \leq \operatorname{lin}(J)}\{\operatorname{reg} J^j - dj\}$*. Then, for* $n \gg 0$*, we have*

$$\operatorname{reg}(I+J)^n = \begin{cases} c(n+1) + f + e^* - 1 & \text{if } c > d, \\ d(n+1) + \max\{e^* + f, e + f^*\} - 1 & \text{if } c = d. \end{cases}$$

Symbolic powers of ideals

Definition

Let R be a commutative ring with identify, and let $I \subseteq R$ be a proper ideal. The n-th *symbolic power* of I is defined to be

$$I^{(n)} := R \cap \Big(\bigcap_{\mathfrak{p} \in \mathrm{Min}_R(R/I)} I^n R_{\mathfrak{p}} \Big).$$

Example

1. If $I = \wp_1 \cap \dots \cap \wp_s$ is the defining ideal of s points in $\mathbb{A}^n_k$ then
$$I^{(n)} = \wp_1^n \cap \dots \cap \wp_s^n.$$
2. If I is a squarefree monomial ideal, $I = \bigcap_{\wp \in \mathrm{Ass}(R/I)} \wp$, then
$$I^{(n)} = \bigcap_{\wp \in \mathrm{Ass}(R/I)} \wp^n.$$

Symbolic powers of ideals

Definition

Let R be a commutative ring with identify, and let $I \subseteq R$ be a proper ideal. The n-th *symbolic power* of I is defined to be

$$I^{(n)} := R \cap \Big(\bigcap_{\mathfrak{p} \in \mathrm{Min}_R(R/I)} I^n R_{\mathfrak{p}} \Big).$$

Example

1. If $I = \wp_1 \cap \cdots \cap \wp_s$ is the defining ideal of s points in $\mathbb{A}^n_k$ then
$$I^{(n)} = \wp_1^n \cap \cdots \cap \wp_s^n.$$
2. If I is a squarefree monomial ideal, $I = \bigcap_{\wp \in \mathrm{Ass}(R/I)} \wp$, then
$$I^{(n)} = \bigcap_{\wp \in \mathrm{Ass}(R/I)} \wp^n.$$

Binomial expansion for symbolic powers

- $A = k[x_1, \ldots, x_r]$, $B = k[y_1, \ldots, y_s]$ are polynomial rings.
- $I \subseteq A$ and $J \subseteq B$ are nonzero proper homogeneous ideals.
- $R = A \otimes_k B = k[x_1, \ldots, x_r, y_1, \ldots, y_s]$.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For all $n \geq 1$, we have

$$(I + J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}.$$

- This expansion was recently proved for *squarefree monomial ideals* by **Bocci, Cooper, Guardo, Harbourne, Janssen, Nagel, Seceleanu, Van Tuyl,** and **Vu.**

Binomial expansion for symbolic powers

- $A = k[x_1, \ldots, x_r]$, $B = k[y_1, \ldots, y_s]$ are polynomial rings.
- $I \subseteq A$ and $J \subseteq B$ are nonzero proper homogeneous ideals.
- $R = A \otimes_k B = k[x_1, \ldots, x_r, y_1, \ldots, y_s]$.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For all $n \geq 1$, we have

$$(I + J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}.$$

- This expansion was recently proved for *squarefree monomial ideals* by **Bocci, Cooper, Guardo, Harbourne, Janssen, Nagel, Seceleanu, Van Tuyl,** and **Vu.**

Corollary

$(I+J)^{(n)} = (I+J)^n$ if and only if $I^{(t)} = I^t$ and $J^{(t)} = J^t$ for all $t \le n$.

Definition

For a homogeneous ideal K set $\alpha(K) := \min\{d \mid K_d \neq 0\}$. The *Waldschmidt constant* of K is defined to be

$$\hat{\alpha}(K) := \lim_{n\to\infty} \frac{\alpha(K^{(n)})}{n}.$$

Corollary

$\hat{\alpha}(I+J) = \min\{\hat{\alpha}(I), \hat{\alpha}(J)\}.$

Corollaries

Corollary

$(I+J)^{(n)} = (I+J)^n$ if and only if $I^{(t)} = I^t$ and $J^{(t)} = J^t$ for all $t \leq n$.

Definition

For a homogeneous ideal K set $\alpha(K) := \min\{d \mid K_d \neq 0\}$. The *Waldschmidt constant* of K is defined to be

$$\hat{\alpha}(K) := \lim_{n\to\infty} \frac{\alpha(K^{(n)})}{n}.$$

Corollary

$\hat{\alpha}(I+J) = \min\{\hat{\alpha}(I), \hat{\alpha}(J)\}.$

Corollaries

Corollary

$(I+J)^{(n)} = (I+J)^n$ if and only if $I^{(t)} = I^t$ and $J^{(t)} = J^t$ for all $t \leq n$.

Definition

For a homogeneous ideal K set $\alpha(K) := \min\{d \mid K_d \neq 0\}$. The *Waldschmidt constant* of K is defined to be

$$\hat{\alpha}(K) := \lim_{n\to\infty} \frac{\alpha(K^{(n)})}{n}.$$

Corollary

$\hat{\alpha}(I+J) = \min\{\hat{\alpha}(I), \hat{\alpha}(J)\}$.

Depth, regularity of symbolic powers by approximation

- Set $Q_p := \sum_{t=0}^{p} I^{(n-t)} J^{(t)}$. Then

$$I^{(n)} = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I+J)^{(n)}.$$

- $Q_p / Q_{p-1} = I^{(n-p)} J^{(p)} / I^{(n-p+1)} J^{(p)}$.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For $n \geq 1$, we have

1. depth $R/(I+J)^{(n)} \geq$
$$\min_{i \in [1,n-1],\, j \in [1,n]} \left\{ \operatorname{depth} A/I^{(n-i)} + \operatorname{depth} B/J^{(i)} + 1, \operatorname{depth} A/I^{(n-j+1)} + \operatorname{depth} B/J^{(j)} \right\}.$$
2. reg $R/(I+J)^{(n)} \leq$
$$\max_{i \in [1,n-1],\, j \in [1,n]} \left\{ \operatorname{reg} A/I^{(n-i)} + \operatorname{reg} B/J^{(i)} + 1, \operatorname{reg} A/I^{(n-j+1)} + \operatorname{reg} B/J^{(j)} \right\}.$$

Moreover, if either char $k = 0$ or I and J are monomial ideals then we have the equalities.

- Set $Q_p := \sum_{t=0}^{p} I^{(n-t)}J^{(t)}$. Then
$$I^{(n)} = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I+J)^{(n)}.$$
- $Q_p/Q_{p-1} = I^{(n-p)}J^{(p)}/I^{(n-p+1)}J^{(p)}$.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For $n \geq 1$, we have

1. depth $R/(I+J)^{(n)} \geq$
$$\min_{i\in[1,n-1],\ j\in[1,n]} \left\{ \operatorname{depth} A/I^{(n-i)} + \operatorname{depth} B/J^{(i)} + 1, \operatorname{depth} A/I^{(n-j+1)} + \operatorname{depth} B/J^{(j)} \right\}.$$
2. reg $R/(I+J)^{(n)} \leq$
$$\max_{i\in[1,n-1],\ j\in[1,n]} \left\{ \operatorname{reg} A/I^{(n-i)} + \operatorname{reg} B/J^{(i)} + 1, \operatorname{reg} A/I^{(n-j+1)} + \operatorname{reg} B/J^{(j)} \right\}.$$

Moreover, if either char $k = 0$ or I and J are monomial ideals then we have the equalities.

- Set $Q_p := \sum_{t=0}^{p} I^{(n-t)}J^{(t)}$. Then
$$I^{(n)} = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I+J)^{(n)}.$$
- $Q_p/Q_{p-1} = I^{(n-p)}J^{(p)}/I^{(n-p+1)}J^{(p)}$.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For $n \geq 1$, we have

1. depth $R/(I+J)^{(n)} \geq$
$$\min_{i\in[1,n-1],\ j\in[1,n]} \left\{ \operatorname{depth} A/I^{(n-i)} + \operatorname{depth} B/J^{(i)} + 1, \ \operatorname{depth} A/I^{(n-j+1)} + \operatorname{depth} B/J^{(j)} \right\}.$$
2. reg $R/(I+J)^{(n)} \leq$
$$\max_{i\in[1,n-1],\ j\in[1,n]} \left\{ \operatorname{reg} A/I^{(n-i)} + \operatorname{reg} B/J^{(i)} + 1, \ \operatorname{reg} A/I^{(n-j+1)} + \operatorname{reg} B/J^{(j)} \right\}.$$

Moreover, if either char $k = 0$ or I and J are monomial ideals then we have the equalities.

Corollary

Assume that J is generated by variables. Then

1. depth $R/(I+J)^{(n)} = \min_{i \leq n} \operatorname{depth} A/I^{(i)} + \dim B/J$; *and*
2. reg $R/(I+J)^{(n)} = \max_{i \leq n}\{\operatorname{reg} A/I^{(i)} - i\} + n$.

Proposition

$$(I+J)^{(n)}/(I+J)^{(n+1)} = \bigoplus_{i+j=n} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$$

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For all $n \geq 1$, we have

1. $\operatorname{depth} \dfrac{(I+J)^{(n)}}{(I+J)^{(n+1)}} = \min_{i+j=n} \left\{ \operatorname{depth} \dfrac{I^{(i)}}{I^{(i+1)}} + \operatorname{depth} \dfrac{J^{(j)}}{J^{(j+1)}} \right\}.$
2. $\operatorname{reg} \dfrac{(I+J)^{(n)}}{(I+J)^{(n+1)}} = \max_{i+j=n} \left\{ \operatorname{reg} I^{(i)}/I^{(i+1)} + \operatorname{reg} J^{(j)}/J^{(j+1)} \right\}.$

Proposition

$$(I+J)^{(n)}/(I+J)^{(n+1)} = \bigoplus_{i+j=n} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$$

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

For all $n \geq 1$, we have

1. $$\operatorname{depth} \frac{(I+J)^{(n)}}{(I+J)^{(n+1)}} = \min_{i+j=n} \left\{ \operatorname{depth} \frac{I^{(i)}}{I^{(i+1)}} + \operatorname{depth} \frac{J^{(j)}}{J^{(j+1)}} \right\}.$$
2. $$\operatorname{reg} \frac{(I+J)^{(n)}}{(I+J)^{(n+1)}} = \max_{i+j=n} \left\{ \operatorname{reg} I^{(i)}/I^{(i+1)} + \operatorname{reg} J^{(j)}/J^{(j+1)} \right\}.$$

Cohen-Macaulayness of symbolic powers

Corollary

The following are equivalent:

1. $R/(I+J)^{(t)}$ is Cohen-Macaulay for all $t \leq n$;
2. $(I+J)^{(n-1)}/(I+J)^{(n)}$ is Cohen-Macaulay;
3. $A/I^{(t)}$ and $B/J^{(t)}$ are Cohen-Macaulay for all $t \leq n$;
4. $I^{(t)}/I^{(t+1)}$ and $J^{(t)}/J^{(t+1)}$ are Cohen-Macaulay for all $t \leq n-1$.

Proof of the binomial expansion

How to prove the binomial expansion

$$(I+J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}?$$

- Let $S_n = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}$.
- $S_n \subseteq (I+J)^{(n)}$.
- Consider the short exact sequences

$$0 \longrightarrow S_{p-1}/S_p \longrightarrow R/S_p \longrightarrow R/S_{p-1} \longrightarrow 0$$

  to get

$$\operatorname{Ass}_R(R/S_n) = \bigcup_{p=1}^{n} \operatorname{Ass}_R(S_{p-1}/S_p).$$

- $S_{p-1}/S_p = \bigoplus_{i+j=p-1} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)} \right).$

Proof of the binomial expansion

How to prove the binomial expansion

$$(I+J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}?$$

- Let $S_n = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}$.
- $S_n \subseteq (I+J)^{(n)}$.
- Consider the short exact sequences

$$0 \longrightarrow S_{p-1}/S_p \longrightarrow R/S_p \longrightarrow R/S_{p-1} \longrightarrow 0$$

to get

$$\mathrm{Ass}_R(R/S_n) = \bigcup_{p=1}^{n} \mathrm{Ass}_R(S_{p-1}/S_p).$$

- $S_{p-1}/S_p = \bigoplus_{i+j=p-1} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$

Proof of the binomial expansion

How to prove the binomial expansion

$$(I+J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}?$$

- Let $S_n = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}$.
- $S_n \subseteq (I+J)^{(n)}$.
- Consider the short exact sequences

$$0 \longrightarrow S_{p-1}/S_p \longrightarrow R/S_p \longrightarrow R/S_{p-1} \longrightarrow 0$$

  to get

$$\mathrm{Ass}_R(R/S_n) = \bigcup_{p=1}^{n} \mathrm{Ass}_R(S_{p-1}/S_p).$$

- $S_{p-1}/S_p = \bigoplus_{i+j=p-1} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)} \right).$

Proof of the binomial expansion

How to prove the binomial expansion

$$(I+J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}?$$

- Let $S_n = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}$.
- $S_n \subseteq (I+J)^{(n)}$.
- Consider the short exact sequences

$$0 \longrightarrow S_{p-1}/S_p \longrightarrow R/S_p \longrightarrow R/S_{p-1} \longrightarrow 0$$

to get

$$\mathrm{Ass}_R(R/S_n) = \bigcup_{p=1}^{n} \mathrm{Ass}_R(S_{p-1}/S_p).$$

- $S_{p-1}/S_p = \bigoplus_{i+j=p-1} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$

Proof of the binomial expansion

How to prove the binomial expansion

$$(I+J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}?$$

- Let $S_n = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}$.
- $S_n \subseteq (I+J)^{(n)}$.
- Consider the short exact sequences

$$0 \longrightarrow S_{p-1}/S_p \longrightarrow R/S_p \longrightarrow R/S_{p-1} \longrightarrow 0$$

  to get

$$\operatorname{Ass}_R(R/S_n) = \bigcup_{p=1}^{n} \operatorname{Ass}_R(S_{p-1}/S_p).$$

- $S_{p-1}/S_p = \bigoplus_{i+j=p-1} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$

Problem

Let M and N be nonzero finitely generated modules over A and B, respectively. Describe the associated primes of the R-module $M \otimes_k N$ in terms of the associated primes of M and N.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

Let $\mathrm{Ass}_{-}(-)$ *and* $\mathrm{Min}_{-}(-)$ *denote the set of associated and minimal primes. Then*

1. $\mathrm{Min}_R(M \otimes_k N) = \bigcup_{\mathfrak{p} \in \mathrm{Min}_A(M), \mathfrak{q} \in \mathrm{Min}_B(N)} \mathrm{Min}_R(R/\mathfrak{p} + \mathfrak{q}).$
2. $\mathrm{Ass}_R(M \otimes_k N) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M), \mathfrak{q} \in \mathrm{Ass}_B(N)} \mathrm{Min}_R(R/\mathfrak{p} + \mathfrak{q}).$

Problem

Let M and N be nonzero finitely generated modules over A and B, respectively. Describe the associated primes of the R-module $M \otimes_k N$ in terms of the associated primes of M and N.

Theorem (—, H.D. Nguyen, N.V. Trung and T.N. Trung)

Let $\mathrm{Ass}_-(-)$ *and* $\mathrm{Min}_-(-)$ *denote the set of associated and minimal primes. Then*

1. $$\mathrm{Min}_R(M \otimes_k N) = \bigcup_{\mathfrak{p} \in \mathrm{Min}_A(M), \mathfrak{q} \in \mathrm{Min}_B(N)} \mathrm{Min}_R(R/\mathfrak{p} + \mathfrak{q}).$$
2. $$\mathrm{Ass}_R(M \otimes_k N) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M), \mathfrak{q} \in \mathrm{Ass}_B(N)} \mathrm{Min}_R(R/\mathfrak{p} + \mathfrak{q}).$$


Thank You!