Lattice simplices with a given degree
RIMS 2016

Johannes Hofscheier
Otto-von-Guericke-Universität Magdeburg

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Outline

1. Motivation

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Definitions

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$\exists\ \mathrm{Ehr}_P \in \mathbb{Q}[t]$ (*Ehrhart polynomial*) s.t. $\mathrm{Ehr}_P(k) := \left|k\Delta \cap \mathbb{Z}^d\right|$ $(k \in \mathbb{Z})$.

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degree of P: $\deg(P) = \deg h_P^*(t)$.

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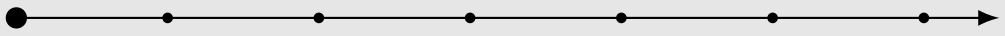
$P = [0, a] \subseteq \mathbb{R}$ $(a \in \mathbb{Z})$ 

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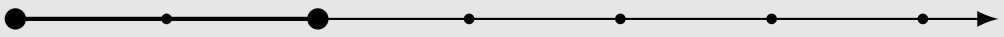
$P = [0, a] \subseteq \mathbb{R} \ (a \in \mathbb{Z})$ 

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Can one characterize all polynomials which can be interpreted as the h^*-polynomial of some lattice polytope?

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All lin. polynomials $1 + at \in \mathbb{Z}^2_{\geq 0}[t]$ can be interpreted as h^*-vectors.

Degree 2[Henk & Tagami, Treutlein]

All polynomials $1 + a_1 t + a_2 t^2 \in \mathbb{Z}_{\geq 0}[t]$ with

$$a_1 \leq \begin{cases} 7 & \text{if } a_2 = 1 \\ 3a_2 + 3 & \text{if } a_2 \geq 2 \end{cases}$$

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Interpret $h^* = 1 + a_1 t + a_2 t^2 \in \mathbb{Z}_{\geq 0}[t]$ as point in the positive orthant $(a_1, a_2) \in \mathbb{R}^2_{\geq 0}$.

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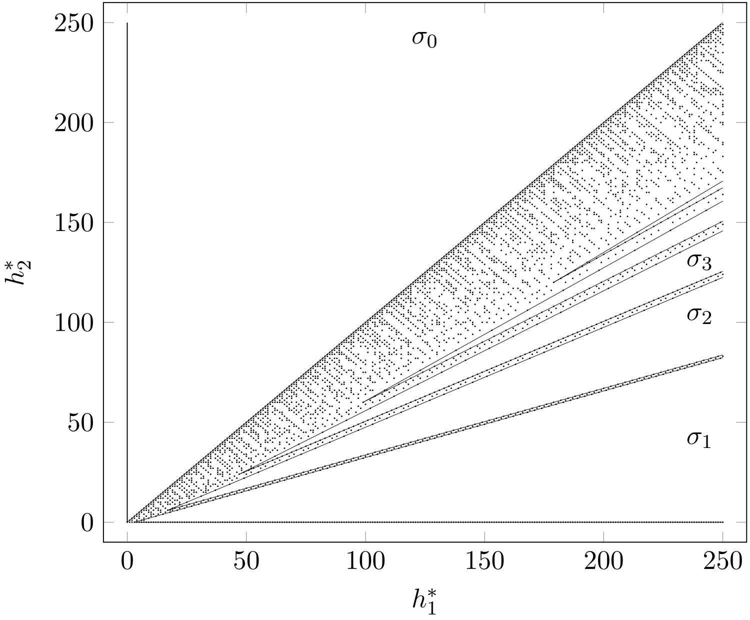
Proposition[H.,Nill,Oeberg]

There is a family $(\sigma_i)_{i \in \mathbb{Z}_{\geq 0}}$ of affine cones $\sigma_i \subseteq \mathbb{R}^2_{\geq 0}$ such that $\mathcal{M} \cap \sigma_i^\circ = \emptyset$ for all i.

Degree 2



250
200
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100
50
0
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σ_0
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Idea

Question C $\Rightarrow$ Question B.

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- Vertices stay the same: What is a good choice?

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- $\Lambda_\Delta := \varphi(\mathbb{Z}^{d+1}) \subseteq \mathbb{R}^d$ lattice

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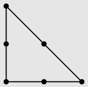

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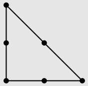

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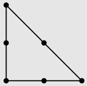

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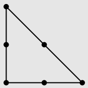

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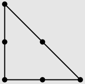

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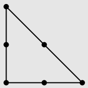

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Idea of Proof

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Properties of Λ_Δ

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$\Delta = \operatorname{conv}(\mathbf{v}_1, \dots, \mathbf{v}_{d+1}) \subseteq \mathbb{R}^d$ d-dimensional lattice simplex.
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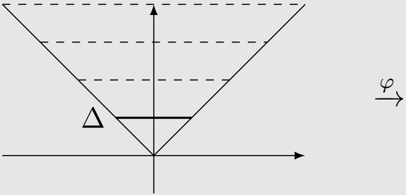
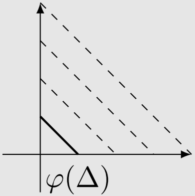
2. 



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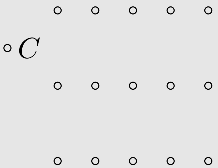

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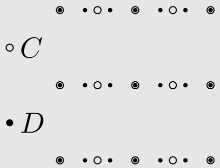

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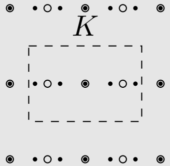

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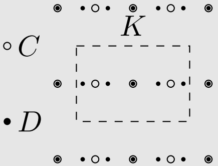

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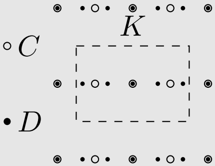

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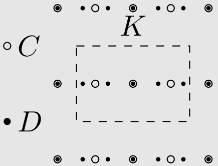

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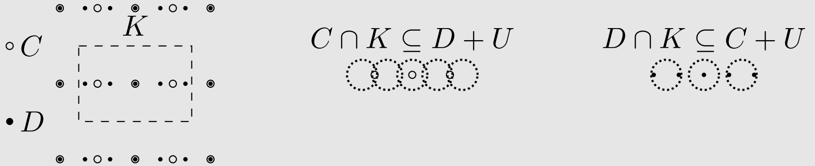

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Theorem

The set of maximal elements in $\mathscr{C}_{\leq s}$ is finite.

Idea of Proof

Set $\mathcal{M} := \{C \in \mathscr{C}_{\leq s} \text{ maximal}\}$.

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Chabauty-Pontryagin Duality[Cornulier]

The duality map

$$* \colon \left\{\text{cl. subgrp. } \subseteq \mathbb{R}^{d+1}\right\} \to \left\{\text{cl. subgrp. } \subseteq \mathbb{R}^{d+1}\right\};$$

$$C \mapsto C^* := \left\{\mathbf{x} \in \mathbb{R}^{d+1} \colon \sum_{i=1}^{d+1} x_i y_i \in \mathbb{Z} \ \forall \mathbf{y} \in C\right\}$$

is an involutory homeomorphism.

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Remark

Could be also proved using a result due to Lawrence.

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Let $\Delta \subseteq \mathbb{R}^d$ be a lattice polytope. *Lattice pyramid* over Δ:

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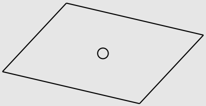

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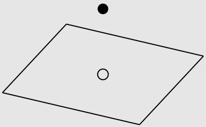

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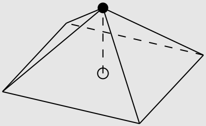

Lattice Pyramids

Definition

Let $\Delta \subseteq \mathbb{R}^d$ be a lattice polytope. *Lattice pyramid* over Δ:

$$\operatorname{conv}(\Delta \times \{0\}, \mathbf{e}_{d+1}) \subseteq \mathbb{R}^{d+1}.$$

Example

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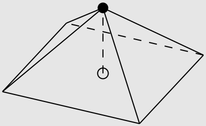

Proposition

The d-dim. lattice simplex $\Delta \subseteq \mathbb{R}^d$ is a lattice pyramid iff $\pi_i(\Lambda_\Delta) = \mathbb{Z}$ where $\pi_i \colon \mathbb{R}^{d+1} \to \mathbb{R}; \mathbf{x} \mapsto x_i$ for $i = 1, \ldots, d+1$.

Simplices of given Degree

Theorem[Nill]

Let $\Delta \subseteq \mathbb{R}^d$ be a d-dim. lattice simplex. If $d \geq 4\deg(\Delta) - 1$, then Δ is a lattice pyramid.

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Corollary

The assignment $\Delta \mapsto \Lambda_\Delta$ induces a bijection

$$\left\{\begin{matrix}\text{lattice simplices } \Delta \\ \text{with } \deg(\Delta) \leq s\end{matrix}\right\} / \sim_1 \leftrightarrow (\mathscr{C}_{\leq s} \cap \mathcal{D}) / \sim_2$$

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- $\sim_2 =$ up to permutation of the coordinates.

Degree 1

Theorem [Batyrev, Nill]

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Describe all lattice simplices of degree at most 1.

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Describe the maximal elements in $\mathscr{C}_{\leq 1} \cap \mathcal{D} \subseteq \mathbb{R}^3$.

Corollary

The maximal elements in $\mathscr{C}_{\leq 1} \cap \mathcal{D} \subseteq \mathbb{R}^3$ are the following:

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{and} \qquad \mathbb{Z}^3 + \mathbb{R}(\mathbf{e}_1 - \mathbf{e}_2) =: \left(\overline{1 \; -1 \; 0} \right)$$

Degree 2 case

Recall: $\mathbf{a}_1, \ldots, \mathbf{a}_p, \mathbf{b}_1, \ldots, \mathbf{b}_q \in \mathbb{R}^{d+1}$ linearly indep.

$$\begin{pmatrix} - \ \mathbf{a}_1 \ - \\ \vdots \\ - \ \mathbf{a}_p \ - \\ \hline - \ \mathbf{b}_1 \ - \\ \vdots \\ - \ \mathbf{b}_q \ - \end{pmatrix} := \bigoplus_{i=1}^{p} \mathbb{Z}\mathbf{a}_i \oplus \bigoplus_{j=1}^{q} \mathbb{R}\mathbf{b}_j$$

Degree 2 case (continued)

Theorem[Higashitani, H.]

The maximal elements in $\mathscr{C}_{\leq 2} \subseteq \mathbb{R}^7$ are the following:

1. $\left(\begin{array}{cccccc} \hline -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$
2. $\left(\begin{array}{cccccc} \hline 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right)$ $\left(\begin{array}{cccccc} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right)$ $\left(\begin{array}{cccccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \hline 1 & 1 & -2 & 0 & 0 & 0 \end{array}\right)$ $+9$ more

   discr. subgrp.
3. $\left(\begin{array}{cccccc} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$
4. All other max. subgrp. are discrete and $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^{d+1}$.

Cayley Conjecture

Definition

$\Delta \subseteq \mathbb{R}^d$ is a *Cayley Polytope* of lattice polytopes $\Delta_1, \ldots, \Delta_k \subseteq \mathbb{R}^m$ if $k \geq 2$ and Δ is unimodularly equivalent to $\mathrm{conv}(\Delta_1 \times \mathbf{e}_1, \ldots, \Delta_k \times \mathbf{e}_k) \subseteq \mathbb{R}^m \times \mathbb{R}^k$.

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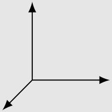
Example

$\Delta_1, \Delta_2 \subseteq \mathbb{R}$ two (lattice) intervals.



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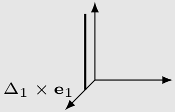

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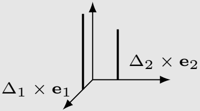

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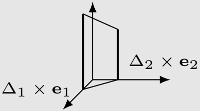

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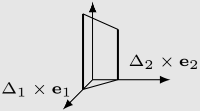

"Weak" Cayley Conjecture[Dickenstein, Nill]

A d-dim. lattice polytpe with degree s is a Cayley polytope, if $d > 2s$.

Cayley Conjecture

Proposition[Higashitani,H.]

The “Weak” Cayley Conjecture holds for degree 2 simplices.

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Let $\Lambda \in \mathscr{C} \cap \mathcal{D} \subseteq \mathbb{R}^{d+1}$.

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Let $\Lambda \in \mathscr{C} \cap \mathcal{D} \subseteq \mathbb{R}^{d+1}$. Λ corresponds to a Cayley polytope iff there is a proper subset $I \subsetneq \{1, \ldots, d+1\}$ such that $f_I(\Lambda) \subseteq \mathbb{Z}$ where $f_I : \Lambda \to \mathbb{R}; \mathbf{x} \mapsto \sum_{i \in I} x_i$.

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Need to consider the case of dim. at least 5, i. e., the max. subgrp. satisfy $\Lambda \subseteq \frac{1}{2}\mathbb{Z}^7$. Example ($6$-dim.):

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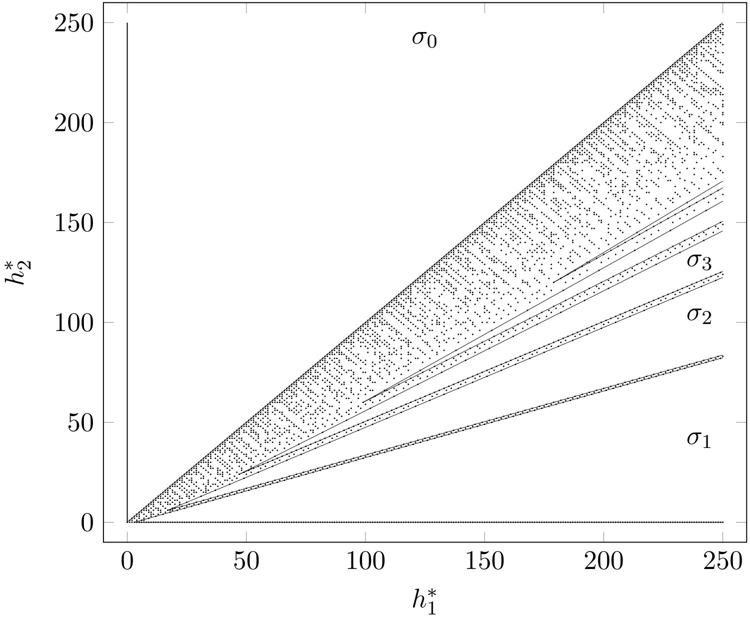
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

for instance $I = \{1, 2, 3, 4\}$. □

Not realizable h^*-vectors



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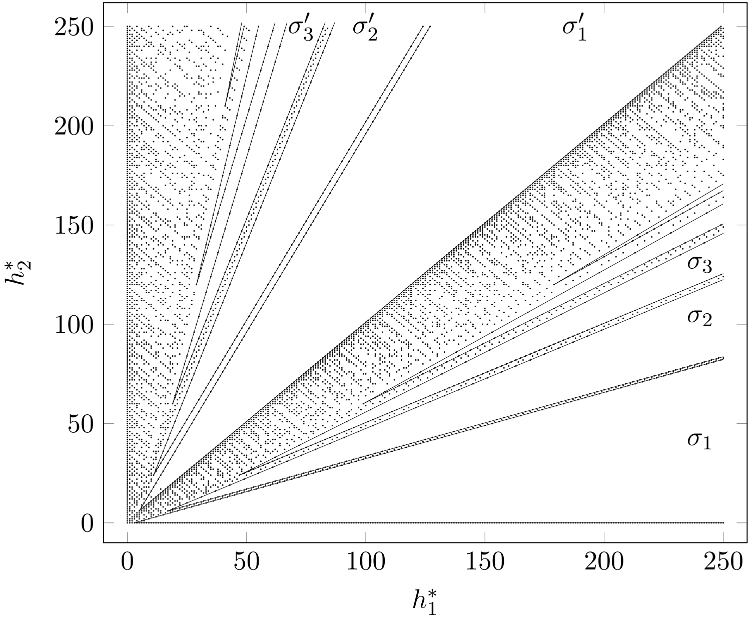

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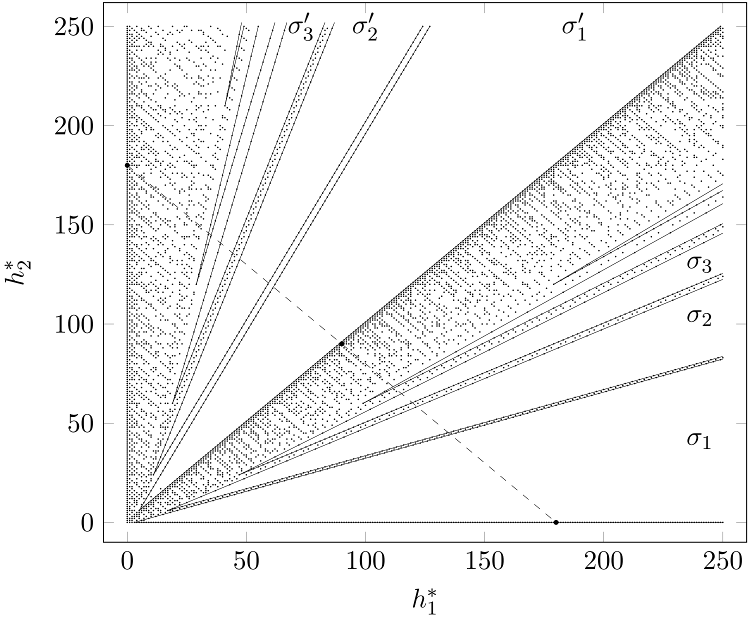

Thank you!