Multiplicative structures on minimal free resolutions of monomial ideals

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Section 1

General setting

Notation and conventions

- $\mathbb{k}$: some field.
- $S = \mathbb{k}[x_1, \dots, x_n]$: polynomial ring over $\mathbb{k}$, either with the standard grading or with the fine $\mathbb{Z}^n$-grading.
- $I \subset S$: homogeneous ideal

Reminder: Minimal free resolutions

A minimal free resolution of S/I is an exact sequence of free S-modules

$$F_\bullet : \qquad 0 \to F_p \xrightarrow{\partial} F_{p-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0$$

such that $F_0/\partial(F_1) \cong S/I$ and $\partial(F_i) \subset \mathfrak{m}F_{i-1}$ where $\mathfrak{m} = (x_1, \ldots, x_n)$.

Some facts:

- A minimal free resolution always exists and it is unique up to isomorphism. So we may speak about "the" minimal free resolution of S/I.
- All maps in $F_\bullet$ can be chosen homogeneous with respect to the grading from S.
- We consider $F_\bullet$ as the free S-module $\bigoplus_i F_i$. Note that we have two gradings: The homological one, and the one coming from S.

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More notation and conventions

- $F_\bullet$: minimal free resolution of S/I.
- For $g \in F_\bullet$:
  - $|g|$: homological degree,
  - $\deg g$: degree with respect to the grading on S.
- $\beta_i(S/I) := \operatorname{rank} F_i$: ith Betti number of S/I.

Remark: $F_0 \cong S$, so $\beta_0(S/I) = 1$. Further, $\beta_1(S/I)$ is the minimal number of generators of I.

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Multiplications on free resolutions

General idea: Turn the whole resolution into an algebra over S.

Definition

A multiplication on $F_\bullet$ is a map $\cdot : F_\bullet \otimes_S F_\bullet \to F_\bullet$ which is

1. S-linear and respects both the homological and the internal grading,
2. satisfies a Leibniz rule: $\partial(a \cdot b) = (\partial a) \cdot b + (-1)^{|a|} a \cdot \partial b$.
3. and extends the multiplication on $F_0 = S$.

Definition

A DGA structure on $F_\bullet$ is a multiplication, which in addition is

4. graded-commutative: $a \cdot b = (-1)^{|a| \cdot |b|} b \cdot a$ and
5. associative.

DGA: Differential graded algebra.

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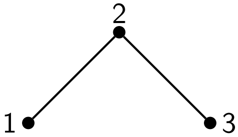

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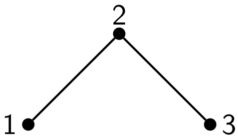

Multiplication on $F_\bullet$?

Existence of multiplications: Motivation

Why do we care about multiplications?

DGA Structures on $F_\bullet$ allow us to infer statements about $F_\bullet$. For example:

Proposition (Eisenbud-Buchsbaum '77)

If $F_\bullet$ admits a DGA structure, then

$$\beta_i(S/I) \geq \binom{g}{i}$$

for $0 \leq i \leq g$, where $g = \mathrm{grade}\, I$.

Recall: $\mathrm{grade}\, I$ is the maximal length of a regular sequence in I.

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Proposition

Every resolution of S/I admits a multiplication, and it is unique up to homotopy.

Proof: Comparison Theorem (or Homotopy Transfer Theorem).

- "up to homotopy": If μ, μ' are two multiplications, then there exists a map $\sigma : F_\bullet \otimes F_\bullet \to F_\bullet$ of homological degree $+1$, such that $\mu - \mu' = \sigma \circ \partial + \partial \circ \sigma$.
- (Eisenbud-Buchsbaum '77) There always exists a multiplication which is graded-commutative.

From now on: All multiplications are assumed to be graded-commutative.

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- (Buchsbaum-Eisenbud '77) Conjecture: One can always find a DGA structure on $F_\bullet$.
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Remark: For a given ideal $I \subset S$, one can always find a non-minimal free resolution which does admit a DGA structure.

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An ideal without a DGA resolution

Consider the ideal $I \subset \mathbb{k}[x, y, z, v]$ with generators

$$m_1 = x^2,\ m_2 = xy,\ m_3 = y^2z^2,\ m_4 = zv,\ m_5 = v^2$$

This ideal is generic, so its minimal free resolution is supported on its Scarf complex:

It does not admit a DGA structure. More precisely, it does not admit a DGA-structure which respects the multigrading.

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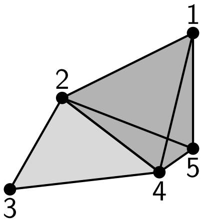

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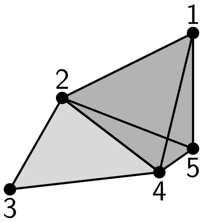

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Existence of multiplications

There are several cases where the minimal free resolution is known to carry a DGA structure:

- complete intersections (Koszul complex),
- pdim $S/I \leq 3$ (Buchsbaum-Eisenbud '77)
- pdim $S/I \leq 4$ and Gorenstein (Kustin-Miller '80)
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A little surprisingly, it is enough to modify the last map in $F_\bullet$ to ensure that there is a DGA structure:

Theorem (K. 2016)

Let $I \subset S$ be a homogeneous ideal. Then there exists a homogeneous element $s \in S$, such that the minimal free resolution of sI has a DGA structure.

This holds for regular local rings as well.

Also, if I is multigraded, then s can be chosen homogeneous with respect to the multigrading.

Note that $\operatorname{grade} sI = 1$, so unfortunately this result does not provide lower bounds for Betti numbers.

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Section 2

The monomial case

Monomial ideals

From now on:

- I is a monomial ideal, and
- $F_\bullet$ is multigraded minimal free resolution of S/I.
- Moreover, we only consider multiplications on $F_\bullet$ which respect the multigrading.

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Supportive Multiplications

The lcm-lattice L_I of I is the lattice of all least common multiples of subsets of the generators of I.

- (Gasharov-Peeva-Welker '99) The structure of $F_\bullet$ is determined by the isomorphism type of L_I.

The possible multiplications on $F_\bullet$ are not determined by L_I, because the lcm-lattices of I and sI are isomorphic for every monomial $s \in S$.

Therefore we restrict the class of multiplications we consider:

Definition

We call a multiplication on $F_\bullet$ supportive if for all $a, b \in F_\bullet$, it holds that $a \cdot b = mc$ for a monomial $m \in S$ and a $c \in F_\bullet$ with $\deg c \leq (\deg a) \vee (\deg b)$.

Here, "$\vee$" denotes the componentwise maximum.

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Supportive Multiplications: Example

Consider the ideal I generated by $m_1 = x^2, m_2 = xy$ and $m_3 = xz$.

Its minimal free resolution looks like this:

Possible choices for $g_2 \cdot g_3$:

$$g_2 \cdot g_3 = \begin{cases} xg_{23} & \text{supportive,} \\ -zg_{12} + yg_{13} & \text{not supportive.} \end{cases}$$

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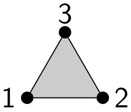

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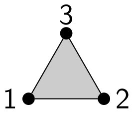

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Supportive Multiplications

Supportive multiplications have desirable properties:

Proposition

1. *If two ideals have isomorphic lcm-lattices, then every supportive multiplication for one ideal induces a supportive multiplication for the other one. Thus, the minimal free resolution of on the ideals admits a DGA structure if and only if the other one does.*
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Proposition (K. 2016)

If $\cdot$ is a supportive multiplication on $F_\bullet$, then for every homogeneous element $a \in F_\bullet$, there exists a monomial $m \in S$ such that

$$ma = \sum_i c_{i,1} \cdot (c_{i,2} \cdot (\cdots (c_{i,r-1} \cdot c_{i,r}) \cdots))$$

for elements $c_{i,j} \in F_1$. In other words, $F_\bullet$ is generated in degree 1, up to multiplication with monomials.

- As a consequence, the differential ∂ on $F_\bullet$ is determined by its part in degree 1 and by the multiplication (Leibniz rule).
- The proposition does not hold for non-suportive multiplications.
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Supportive DGA resolutions and the Taylor resolution

The Taylor resolution is a particular non-minimal DGA resolution. Let us recall the details. For $I = \langle m_1, \dots, m_k \rangle \subset S$ and $N \subseteq [k]$ let $m_N := lcm(m_i \mid i \in N)$.

The Taylor resolution of S/I is the complex

$$T_\bullet : \qquad 0 \to T_k \to T_{k-1} \to \cdots \to T_1 \to S \to S/I \to 0$$

where

$$T_i := \bigoplus_{\substack{N \subseteq [k] \\ \#N = i}} S e_N \qquad \text{and} \qquad \partial e_N := \sum_{i \in N} \pm \frac{m_N}{m_{N \setminus i}} e_{N \setminus i}$$

and $\deg e_N := \deg m_N$.

This is a free DGA resolution of S/I (typically far from being minimal) with the following multiplication:

$$e_N \cdot e_M = \pm \frac{m_N m_M}{m_{N \cup M}} e_{N \cup M}, \text{ if } N \cap M = \emptyset$$

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The Taylor resolution is a non-minimal supportive DGA resolution of S/I.

In fact it is "free" with this properties in the following sense:

Theorem (K. 2016)

Let $I \subset S$ be a monomial ideal with minimal free resolution $F_\bullet$ and Taylor resolution $T_\bullet$. If $F_\bullet$ admits a supportive DGA structure, then $F_\bullet \cong T_\bullet/J$ as DGAs for some DG-ideal $J \subset T_\bullet$.

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Section 3

Applications

Betti vectors

The fact that supportive DGA resolutions are generated in degree 1 allows us to characterize the possible Betti numbers of monomial ideals with such resolutions.

Theorem (K., Welker 2016)

For a vector $b = (b_0, b_1 \dots, b_n) \in \mathbb{N}^n$, the following are equivalent:

1. *There exists a monomial ideal $I \subset S$, whose minimal free resolution is a supportive DGA and $b_i = \beta_i(S/I)$ for all i.*
2. *b is the f-vector of some simplicial complex Δ which is a cone.*
3. *Let $b(t) := \sum_i b_i t^i$. Then $(1-t)$ divides $b(t)$ and*

(Kalai '85) These vectors are also exactly the f-vectors of acyclic simplicial complexes.

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Idea of the proof:

"$1 \Rightarrow 2$": Macaulay-type theorem for DGAs generated in degree 1: (Aramova, Herzog and Hibi '97) Hilbert series of graded-commutative algebras generated in degree 1 are characterized by the Kruskal-Katona inequalities. We modify their proof to show that the existence of the differential restricts the possible Hilbert series to f-vectors of cones.

"$2 \Rightarrow 1$": For a given cone Δ, their exists a monomial ideal whose Scarf complex equals Δ, so in particular its Betti vector equals the f-vector of Δ.
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Does the Betti vector of any monomial ideal satisfy these conditions?
Answer: No.
The Betti vector of $I = (ab, bc, cd, de, ef, fa)$ is $(1, 6, 9, 6, 2)$. This vector is not even the f-vector of any simplicial complex.
Nevertheless, there exist a monomial ideal with the Betti vectors whose minimal free resolution is a DGA, but not supportive.

Subadditivity of syzygies

For a monomial ideal $I \subset S$ let

$$t_i := \max \{ j \mid \beta_{i,j}(S/I) \neq 0 \}$$

the maximal shift in the i-th step of the resolution of S/I.

Definition

We say that subadditivity of syzygies holds for I if

$$t_{a+b} \leq t_a + t_b$$

for all $1 \leq a, b \leq \operatorname{pdim} S/I$ such that $a + b \leq \operatorname{pdim} S/I$.

This does not hold in general for non-monomial ideals (Avramov-Conca-Iyengar '15), but it is open whether every monomial ideal has this property.

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The fact that supportive multiplications are generated in degree one implies (with some work):

Proposition (K. 2016)

The subadditivity of syzygies holds for all monomial ideals whose minimal free resolution admits a supportive DGA structure.

If $F_\bullet$ does not have a supportive DGA resolution, then one can still use these techniques to conclude that $t_{i+1} \leq t_i + t_1$ for all i. This was shown before by Herzog and Srinivasan by other techniques.

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The end.

Thank you