Pattern-Avoiding Polytopes

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Pattern Avoidance

Let $\mathfrak{S}_n$ denote the symmetric group on $1, 2, \ldots, n$, $\pi \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_n$, written as words.

Definition

Say σ contains the pattern π if there is some substring of σ whose elements have the same relative order as those in π. If no such substring exists, then σ avoids the pattern π. If $\Pi \subseteq \mathfrak{S}$, then σ avoids Π if σ avoids every element of Π.

So 526413 does not avoid 132 while 453621 does.

Denote by

$$\mathrm{Av}_n(\Pi) := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \Pi\}$$

the avoidance class of Π.

Pattern Avoidance

A simple yet difficult question: given Π, determine $|\operatorname{Av}_n(\Pi)|$.

If $\pi = a_1 a_2 \dots a_k$, call

$$\pi^r := a_k a_{n-1} \dots a_1$$

the reversal of π and

$$\pi^c := (k - a_1 + 1)(k - a_2 + 1) \dots (k - a_k + 1)$$

the complement of π. Then $|\operatorname{Av}_n(\pi)| = |\operatorname{Av}_n(\pi^r)| = |\operatorname{Av}_n(\pi^c)|$.

Definition

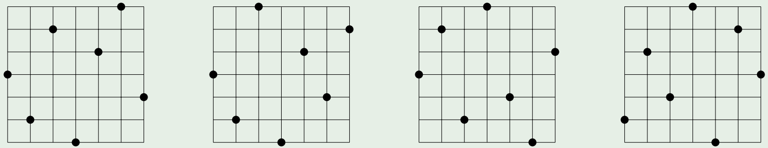
Say π_1 and π_2 are Wilf equivalent, written $\pi_1 \equiv \pi_2$, if $|\operatorname{Av}_n(\pi_1)| = |\operatorname{Av}_n(\pi_2)|$ for all n.

Wilf equivalence *is* an equivalence relation on $\mathfrak{S}$.

Pattern Avoidance

So $\pi \equiv \pi^r \equiv \pi^c$. In fact, π is Wilf equivalent to any permutation obtained by acting on its diagram by the dihedral group of the square. These are called the trivial Wilf equivalences.

Example



$4261573 \equiv 4271536 \equiv 4627315 \equiv 2537164$

Pattern Avoidance

Theorem (MacMahon (1915) and Knuth (1968))

If $\pi \in \mathfrak{S}_3$*, then for all* n*,* $|\operatorname{Av}_n(\pi)| = C_n$*, the* n^{th} *Catalan number.*

Theorem (Erdős-Szekeres (1935))

For any positive integers a, b*, every permutation of length at least* $(a-1)(b-1)+1$ *contains the patterns* $123\ldots a$ *or* $b(b-1)(b-2)\ldots 1$*.*

Theorem (Billey, Burdzy, and Sagan (2012))

For all n*,* $|\operatorname{Av}_n(132, 312)| = 2^{n-1}$*.*

Why study pattern avoidance?

- Stack-sortable permutations
  - A permutation is stack-sortable if and only if it avoids 231 (Knuth, 1968)
- Permutation statistics
  - Almost all known Mahonian permutation statistics really belong to a class of 14 statistics, if the use of vincular patterns is allowed (Babson and Steingrímsson, 2000)
- Classifying smooth / factorial / Gorenstein Schubert varieties using bivincular patterns (Úlfarsson, 2010)

Ehrhart Theory

Definition

For a lattice polytope $P \subseteq \mathbb{R}^n$, its Ehrhart polynomial is

$$\mathcal{L}_P(m) := |mP \cap \mathbb{R}^n|,$$

and its Ehrhart series is

$$\begin{aligned} E_P(t) &:= \sum_{m \geq 0} \mathcal{L}_P(m) t^m \\ &= \frac{h_P^*(t)}{(1-t)^{\dim P + 1}}. \end{aligned}$$

The numerator $h_P^*(t)$ is the h^*-polynomial of P and its list of coefficients $h^*(P) := (h_0^*, \ldots, h_d^*)$ is the h^*-vector of P.

Two Big Questions

1. When is $h^*(P)$ palindromic?
   - This happens exactly when P is Gorenstein, a property that that is often reasonably detectable if a hyperplane description of P is known.
2. When is $h^*(P)$ unimodal? Various sufficient conditions are known, but necessary conditions are not as clear.

Π-avoiding Permutohedra

Definition

The permutohedron is defined as

$$P_n := \text{conv}\{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 \ldots a_n \in \mathfrak{S}_n\}.$$

Some quick facts about P_n:

1. invariant under the action of $\mathfrak{S}_n$
2. simple zonotope
3. its Ehrhart polynomial is

$$\mathcal{L}_{P_n}(m) = \sum_{i=0}^{n-1} f_i^n m^i,$$

where f_i^n is the number of labeled forests on n vertices with i edges.

Π-avoiding Permutohedra

Definition

For $\Pi \subseteq \mathfrak{S}$, define

$$P_n(\Pi) := \operatorname{conv}\{(a_1, \ldots, a_n) \mid a_1 \ldots a_n \in \operatorname{Av}_n(\Pi)\}$$

to be the Π-avoiding permutohedron.

So if $\Pi = \emptyset$, then $P_n(\Pi) = P_n$.

Important note: this is not a subclass of generalized permutohedra introduced by Postnikov. This fact can be verified by comparing normal fans and using a theorem of Postnikov, Reiner, and Williams.

Π-avoiding Permutohedra

$P_n(\pi)$ is unimodularly equivalent to both $P_n(\pi^r)$ and $P_n(\pi^c)$.
But that's about where it stops.

Example (Trivial Wilf equivalence $\not\Rightarrow$ unimodular equivalence)

Choose $\pi = 1423$ and $\pi' = 2431$. These are related by a 90-degree rotation, but $P_5(\pi)$ has 48 facets while $P_5(\pi')$ only has 46.

Π-avoiding Permutohedra

Theorem (D. and Sagan)

If $\Pi = \{132, 312\}$, *then* $P_n(\Pi)$ *is a rectangular parallelepiped with Ehrhart polynomial*

$$\sum_{i=0}^{n-1} \frac{(n-1)!}{(n-i-1)!} m^i$$

This extends the previous result $|\operatorname{Av}_n(132, 312)| = 2^{n-1}$.

Corollary

The number of interior lattice points of $P_n(132, 312)$ *is the number of derangements of* $\mathfrak{S}_{n-1}$.

(Follows from Ehrhart-Macdonald reciprocity)

Π-avoiding Permutohedra

Theorem (Beck, Jochemko, McCullough, in preparation)

Every lattice zonotope has a unimodal h^*-vector.

Corollary

For all n, $h^*(P_n(132, 312))$ is unimodal.

Π-avoiding Permutohedra

Theorem (D. and Sagan)

If $\Pi = \{123, 132\}$*, then* $P_n(\Pi)$ *is a combinatorial (but not geometric!) cube with Ehrhart polynomial*

$$\frac{m+1}{(n-1)!}\prod_{j=2}^{n-1}(nm+j)$$

($P_n(\Pi)$ is a Pitman-Stanley polytope)

Π-avoiding Permutohedra

Proposition (D. and Sagan)

If $\Pi = \{123, 132, 312\}$, then $P_n(\Pi)$ is a simplex with Ehrhart polynomial $(1+m)^{n-1}$. Hence $h^*_P(t)$ is the Eulerian polynomial $A_{n-1}(t)$.

$P_n(123, 132, 312)$ is (unimodularly equivalent to) the simplex containing certain lecture hall partitions. Work of Corteel, Lee, and Savage imply the Ehrhart-theoretic results (an observation made by Ben Braun).

Π-avoiding Permutohedra

The results for the different avoidance classes were proven in very different ways.

This is common in the world of pattern avoidance.

Π-avoiding Birkhoff Polytopes

Definition

The $n \times n$ Birkhoff polytope is

$$B_n := \mathrm{conv}\{M \in \mathbb{R}^{n\times n} \mid M \text{ a matrix for some } \sigma \in \mathfrak{S}_n\}$$

Some variations:

1. transportation polytopes
2. permutation polytopes (Burggraf, De Loera, Omar)
3. the "symmetric slice" of B_n (Stanley, Jia)

Π-avoiding Birkhoff Polytopes

Definition

For $\Pi \subseteq \mathfrak{S}$, define

$$B_n(\Pi) := \mathrm{conv}\{M \in \mathbb{R}^{n \times n} \mid M \text{ a matrix for some } \sigma \in \mathrm{Av}_n(\Pi)\}$$

to be the Π-avoiding Birkhoff polytope.

This time, if $\pi \in \mathfrak{S}_k$ and π' are trivially Wilf equivalent, then $B_n(\pi)$ and $B_n(\pi')$ **are** unimodularly equivalent.

Alternating permutations

Definition

A permutation $a_1 a_2 \ldots a_n \in \mathfrak{S}_n$ is alternating if $a_1 < a_2 > a_3 < a_4 > a_5 < \cdots$.

Let $\widetilde{\mathrm{Av}}_n(\Pi)$ denote the alternating permutations in $\mathfrak{S}_n$ that avoid Π. Analogously define $\widetilde{B}_n(\Pi)$.

These could also be described as $B_n(\Pi)$ for an appropriate Π if we allow vincular patterns.

Our focus will be on the specific polytopes $B_n(132, 312)$ and $\widetilde{B}_n(123)$.

Π-avoiding Birkhoff Polytopes

Proposition (D. and Sagan)

For all n,

$$\dim B_n(132, 312) = \binom{n}{2}$$

and

$$\dim \widetilde{B}_n(123) = \binom{\lceil n/2 \rceil}{2}$$

Beyond knowing the number of vertices of each, the combinatorial structures of these are completely unknown.

Π-avoiding Birkhoff Polytopes

Theorem (Stanley (1970s), Athanasiadis (2005))

For all n, $h^*(B_n)$ is palindromic and unimodal.

What can we say about $h^*(B_n(\Pi))$?

Main Conjecture

Conjecture (D. and Sagan)

The h^*-vectors of $B_n(132, 312)$ and $\widetilde{B}_n(123)$ are palindromic and unimodal.

Broad strategy:

1. Show that these polytopes have regular, unimodular triangulations
2. Show that these polytopes are Gorenstein

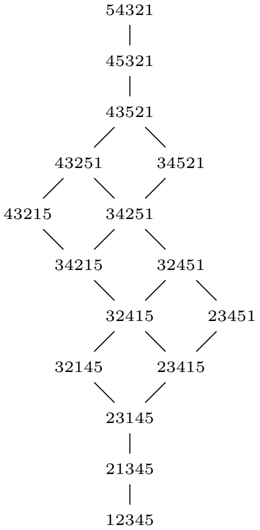
The posets $Q_n(\Pi)$ and $\widetilde{Q}_n(\Pi)$

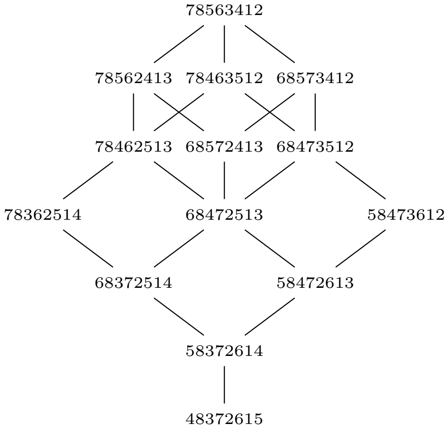
Definition

The right weak (Bruhat) order on $\mathfrak{S}_n$ is defined as $\sigma \lessdot \sigma'$ if $\sigma' = \sigma s_i$ for some simple transposition s_i and σ' has more inversions than σ. The left weak (Bruhat) order is defined analogously.

Let $Q_n(132, 312)$ be the poset on $\mathrm{Av}_n(132, 312)$ induced from the right weak order on $\mathfrak{S}_n$, and $\widetilde{Q}_n(123)$ to be the poset on $\widetilde{\mathrm{Av}}_n(123)$ induced from the left weak order on $\mathfrak{S}_n$.

Examples: $Q_5(132, 312)$ and $\widetilde{Q}_8(123)$





The posets $Q_n(\Pi)$ and $\widetilde{Q}_n(\Pi)$

Theorem (D. and Sagan)

The following isomorphisms hold:

$$Q_n(132, 312) \cong M(n-1),$$

where $M(k)$ is the lattice of shifted Young diagrams with largest part at most k, and

$$\widetilde{Q}_n(123) \cong D^*_{\lceil n/2 \rceil},$$

where D_k is the lattice of Dyck paths of length $2k$ such that if $d_1, d_2 \in D_k$, then $d_1 < d_2$ if d_1 lies entirely underneath d_2.

The posets $Q_n(\Pi)$ and $\widetilde{Q}_n(\Pi)$

From here, we want to use the following facts:

- distributive lattices have EL-labelings
- posets with EL-labelings have shellable order complexes
- given a lattice polytope with a shellable unimodular triangulation, its h^*-vector can be computed based on information about the shelling order

Goal: show that the order complexes of $Q_n(132, 312)$ and $\widetilde{Q}_n(123)$ induce shellable unimodular triangulations of $B_n(132, 312)$ and $\widetilde{B}_n(123)$.

The Commutative Algebra

Conjecture (D. and Sagan)

$B_n(132, 312)$ and $\widetilde{B}_n(123)$ have flag, regular unimodular triangulations.

Theorem (Sturmfels)

For a lattice polytope P, the initial ideals of the toric ideal I_P are in bijection with the regular triangulations of P. The initial ideal of I_P is squarefree if and only if the corresponding triangulation of P is unimodular.

Watermelons, Stars, and Fermi Configurations

Definition

A watermelon $\overline{W}_{l,k}$ is the digraph with vertices

$$\{(-i,-j) \in \mathbb{Z}^2 \mid 0 \leq i \leq l,\ 0 \leq j \leq k,\ j \leq i\}$$

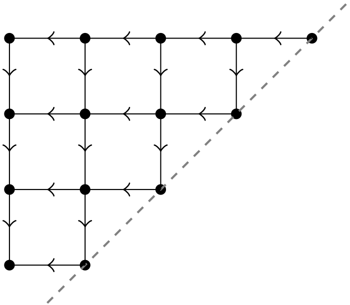
with an arc from a to b if $b - a \in \{-e_1, -e_2\}$. A star graph S_n is the graph whose vertex set is

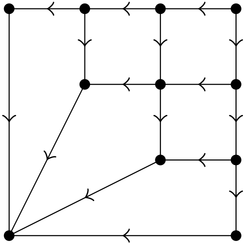
$$\{(-i,-j) \in \mathbb{Z}^2 \mid i,j \geq 0,\ i + j \leq n\}$$

with arcs formed the same way as with watermelons.

To make later definitions simpler, we introduce a unique sink v for S_n by including an arc from the points $(-i, -n+i)$ to v.

Examples: $\overline{W}_{4,3}$ and S_3



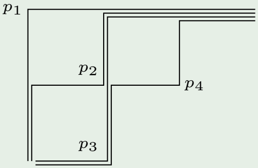


Watermelons, Stars, and Fermi Configurations

Definition

A Fermi configuration in a digraph H with source u and sink v is a collection of distinct, noncrossing paths from u to v. A Fermi configuration is maximal if no additional distinct noncrossing paths from u to v can be included in the configuration.

Example (A maximal Fermi configuration in $\overline{W}_{3,2}$)



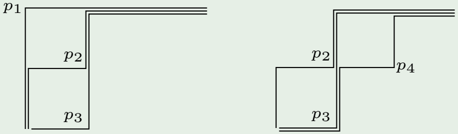
Watermelons, Stars, and Fermi Configurations

Definition

A triple of adjacent paths in a maximal Fermi configuration is called a flipflop if the two 2-dimensional faces it bounds share no edges of the central path. If the central path goes to the right of the first 2-dimensional face it encounters, then the path is called flopped. Otherwise, it is flipped.

Example

The configuration on the previous slide contains the flopped walk (p_2, p_3, p_4) but no flipped walks.



Watermelons, Stars, and Fermi Configurations

Theorem (Arrowsmith, Bhatti, and Essam (2012))

Suppose H is a digraph with unique source and sink, and that H has a unique minimal-cardinality Fermi configuration covering all of its arcs. Let $\varphi_k(H)$ denote the number of maximal Fermi configurations in H that contain k flopped walks. Then the polynomial

$$\Phi(H;t) = \sum_{i \geq 0} \varphi_k(H) t^k$$

has palindromic coefficients.

Watermelons, Stars, and Fermi Configurations

Sagan and I have shown that if the previously-mentioned conjecture holds, then $\Phi(S_n; t)$ is the h^*-polynomial for $B_n(132, 312)$ and $\Phi(\overline{W}_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}; t)$ is the h^*-polynomial for $\widetilde{B}_n(123)$.

It appears that the coefficients of $\Phi(\overline{W}_{k,m}; t)$ are unimodal for all k and m, but it is not immediately obvious how to choose Π so that $\Phi(\overline{W}_{k,m}; t) = B_n(\Pi)$ (or if any such Π exists)

Open Questions

1. Is there a nice combinatorial proof for the number of interior lattice points of $P_n(132, 312)$?
2. For "nice" special classes of Π,
    1. what is the combinatorial structure of $P_n(\Pi)$ or $B_n(\Pi)$?
    2. what is $\mathrm{Vol}(P_n(\Pi))$, $\mathrm{Vol}(B_n(\Pi))$?
    3. what is the Ehrhart polynomial for $P_n(\Pi)$?
    4. what is the h^*-vector of $B_n(\Pi)$?
3. What happens if we consider classes of vincular or bivincular patterns?
4. For which choices of Π is $B_n(\Pi)$ IDP? Gorenstein?
5. What are the homotopy types of $Q_n(\Pi)$? (in general their order complexes aren't necessarily spheres, or even Cohen-Macaulay)