# Characterization of ACM varieties with $d$-linear resolutions

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Kyoto, August 03, 2016,

# Notation and Definition

- $X \subset \mathbb{P}(V) = \mathbb{P}^{n+e}$: a nondegenerate, irreducible and reduced variety (not necessarily smooth) defined over $K = \overline{K}$ of char $(K) \geq 0$, $\dim(X) = n$, $\deg(X) = d$ and $\mathrm{codim}(X, \mathbb{P}(V)) = e$ .
- $R/I_X$: the projective coordinate ring of $X$ where $R = K[x_0, x_1, \ldots, x_{n+e}]$ is a coordinate ring of $\mathbb{P}(V) = \mathbb{P}^{n+e}$, $I_X = \bigoplus_{m \geq 0} H^0(\mathcal{I}_X(m))$ is the saturated ideal.
- $\mathrm{depth}(X) = \mathrm{depth}(R/I_X) = \min\{i \mid H^i(\mathcal{I}_X(m)) \neq 0\}$ for some $m \in \mathbb{Z}, i \geq 1$ and $1 \leq \mathrm{depth}(R/I_X) \leq \dim(R/I_X) = n + 1$.
- $X$ is called ACM if $\mathrm{depth}(R/I_X) = n + 1$, i.e $H^i(\mathbb{P}^{n+e}, \mathcal{I}_X(m)) = 0$ for all $m \in \mathbb{Z}, 1 \leq i \leq n$.
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# Minimal free resolutions

- There are unique minimal free resolutions of $R/I_X$ and $R(X) = \bigoplus_{m \geq 0} H^0(\mathcal{O}_X(m))$ and associated Betti tables.
- It is still interesting to consider the minimal free resolution: $\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0$ where $L_i = \bigoplus_j R(-i-j)^{\beta_{i,j}(X)}$.
- $\beta_{i,j}(X)$ is the rank of the degree $i+j$ part in $L_i$ and $\beta_{i,j}(X) := \dim_K \operatorname{Tor}_i^R(R/I_X, K)_{i+j}$.
- $\beta_{1,1}(X)$: the number of quadrics $Q_i \in I_X$;
- $\beta_{2,1}(X)$ is the number of linear relations of the form $\Sigma L_i Q_i = 0$;
- $\beta_{1,2}(X)$ is the number of cubic generators of $I_X$.

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# Koszul complex and the Betti table

- By the symmetry of Tor, the graded Betti numbers are also defined via the Koszul exact sequence of the base field $K$:
- $V = K\langle x_0, \cdots, x_{n+e}\rangle$ be the $K$-vector space in $K[x_0, \ldots, x_{n+e}]$. Then, $\mathrm{Tor}_i^R(R/I_X, K)_{i+j}$ is the homology of the Koszul complex:

$$\wedge^{i+1} V \otimes (R/I_X)_{j-1} \xrightarrow{\partial_{i+1,j-1}} \wedge^i V \otimes (R/I_X)_j \xrightarrow{\partial_{i,j}} \wedge^{i-1} V \otimes (R/I_X)_{j+1},$$

  where the map is given by $\partial_{i,j}(x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_i} \otimes m) = \sum_{1\leq\mu\leq i} (-1)^{\mu-1} x_{\alpha_1} \cdots \wedge \hat{x_{\alpha_\mu}} \wedge \ldots \wedge x_{\alpha_i} \otimes (x_{\alpha_\mu} \cdot m)$.
- the Koszul complex is exact if $i > n+e+1$ or $j >> 0$ (Hilbert syzygy theorem and Hilbert basis theorem).

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- The Betti table of $R/I_X$

| | 0 | 1 | 2 | 3 | $\cdots$ | $i-1$ | $i$ | $i+1$ | $\cdots$ | $\triangle$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | – | $\cdots$ | – | – | – | $\cdots$ | – |
| 1 | – | $\beta_{1,1}$ | $\beta_{2,1}$ | $\beta_{3,1}$ | $\cdots$ | $\beta_{i-1,1}$ | $\beta_{i,1}$ | $\beta_{i+1,1}$ | $\cdots$ | $\beta_{\triangle,1}$ |
| 2 | – | $\beta_{1,2}$ | $\beta_{2,2}$ | $\beta_{3,2}$ | $\cdots$ | $\beta_{i-1,2}$ | $\beta_{i,2}$ | $\beta_{i+1,2}$ | $\cdots$ | $\beta_{\triangle,2}$ |
| $\vdots$ | – | – | $\dots$ | – | $\ddots$ | $\dots$ | $\vdots$ | $\vdots$ | $\dots$ | $\ddots$ |
| $j$ | – | $\beta_{1,j}$ | $\beta_{2,j}$ | $\beta_{3,j}$ | $\cdots$ | $\beta_{i-1,j}$ | $\beta_{i,j}$ | $\beta_{i+1,j}$ | $\cdots$ | $\beta_{\triangle,j}$ |
| $\vdots$ | | | $\dots$ | – | $\ddots$ | $\dots$ | $\vdots$ | $\vdots$ | $\dots$ | $\ddots$ |
| $\square$ | – | $\beta_{1,\square}$ | $\beta_{2,\square}$ | $\beta_{3,\square}$ | $\cdots$ | $\beta_{i-1,\square}$ | $\beta_{i,\square}$ | $\beta_{i+1,\square}$ | $\cdots$ | $\beta_{\triangle,\square}$ |

- $\triangle =$ the projective dimension of $R/I_X \geq e$.
- $\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1 \leq d - e$ if $X$ is irreducible, reduced (Eisenbud-Goto Conjecture).

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|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | $-$ | $-$ | $-$ | $\cdots$ | $-$ | $-$ | $-$ | $\cdots$ | $-$ |
| 1 | $-$ | $\beta_{1,1}$ | $\beta_{2,1}$ | $\beta_{3,1}$ | $\cdots$ | $\beta_{i-1,1}$ | $\beta_{i,1}$ | $\beta_{i+1,1}$ | $\cdots$ | $\beta_{\triangle,1}$ |
| 2 | $-$ | $\beta_{1,2}$ | $\beta_{2,2}$ | $\beta_{3,2}$ | $\cdots$ | $\beta_{i-1,2}$ | $\beta_{i,2}$ | $\beta_{i+1,2}$ | $\cdots$ | $\beta_{\triangle,2}$ |
| $\vdots$ | $-$ | $-$ | $\ldots$ | $-$ | $\ddots$ | $\ldots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\ddots$ |
| $j$ | $-$ | $\beta_{1,j}$ | $\beta_{2,j}$ | $\beta_{3,j}$ | $\cdots$ | $\beta_{i-1,j}$ | $\beta_{i,j}$ | $\beta_{i+1,j}$ | $\cdots$ | $\beta_{\triangle,j}$ |
| $\vdots$ | | | $\ldots$ | $-$ | $\ddots$ | $\ldots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\ddots$ |
| $\square$ | $-$ | $\beta_{1,\square}$ | $\beta_{2,\square}$ | $\beta_{3,\square}$ | $\cdots$ | $\beta_{i-1,\square}$ | $\beta_{i,\square}$ | $\beta_{i+1,\square}$ | $\cdots$ | $\beta_{\triangle,\square}$ |

- $\triangle =$ the projective dimension of $R/I_X \geq e$.
- $\square = \mathrm{reg}(R/I_X) = \mathrm{reg}(X) - 1 \leq d - e$ if $X$ is irreducible, reduced (Eisenbud-Goto Conjecture).

- The Betti table of $R/I_X$

| | 0 | 1 | 2 | 3 | $\cdots$ | $i-1$ | $i$ | $i+1$ | $\cdots$ | $\triangle$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | $-$ | $-$ | $-$ | $\cdots$ | $-$ | $-$ | $-$ | $\cdots$ | $-$ |
| 1 | $-$ | $\beta_{1,1}$ | $\beta_{2,1}$ | $\beta_{3,1}$ | $\cdots$ | $\beta_{i-1,1}$ | $\beta_{i,1}$ | $\beta_{i+1,1}$ | $\cdots$ | $\beta_{\triangle,1}$ |
| 2 | $-$ | $\beta_{1,2}$ | $\beta_{2,2}$ | $\beta_{3,2}$ | $\cdots$ | $\beta_{i-1,2}$ | $\beta_{i,2}$ | $\beta_{i+1,2}$ | $\cdots$ | $\beta_{\triangle,2}$ |
| $\vdots$ | $-$ | $-$ | $\ldots$ | $-$ | $\ddots$ | $\ldots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\ddots$ |
| $j$ | $-$ | $\beta_{1,j}$ | $\beta_{2,j}$ | $\beta_{3,j}$ | $\cdots$ | $\beta_{i-1,j}$ | $\beta_{i,j}$ | $\beta_{i+1,j}$ | $\cdots$ | $\beta_{\triangle,j}$ |
| $\vdots$ | | | $\ldots$ | $-$ | $\ddots$ | $\ldots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\ddots$ |
| $\square$ | $-$ | $\beta_{1,\square}$ | $\beta_{2,\square}$ | $\beta_{3,\square}$ | $\cdots$ | $\beta_{i-1,\square}$ | $\beta_{i,\square}$ | $\beta_{i+1,\square}$ | $\cdots$ | $\beta_{\triangle,\square}$ |

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- J. Macollough and I. Peeva announced the counterexamples in the seminar talk in the U. of Michigan in July, 2016.
- They claim that the regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the degree. So, it provides counter-examples to the longstanding Eisenbud-Goto Regularity Conjecture.
- For integral curves, EG conjecture is true!(Castelnuovo(1896), Gruson-Lazarsfeld-Peskine (1986)).
- For smooth cases, it is still open and it is good to consider some conditions under which EG conjecture is true even in the integral varieties.

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# The simplest Betti table - Varieties of minimal degree

- $X^n \subset \mathbb{P}^{n+e}$: nondegenerate, irreducible and reduced (not necessarily smooth) of degree $d \geq e+1$. Then $X$ is called a "variety of minimal degree"(VMD) if $d = e+1$.
- The simplest Betti table of $X$ which is 2-linear ACM with

$$\beta_{i,1} = i \cdot \binom{e+1}{i+1}:$$

| | 0 | 1 | 2 | 3 | $\cdots$ | $i-1$ | $i$ | $i+1$ | $\cdots$ | $e$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | – | $\cdots$ | – | – | – | $\cdots$ | – |
| 1 | – | $\beta_{1,1}$ | $\beta_{2,1}$ | $\beta_{3,1}$ | $\cdots$ | $\beta_{i-1,1}$ | $\beta_{i,1}$ | $\beta_{i+1,1}$ | $\cdots$ | $\beta_{e,1}$ |

Table 1 minimal degree varieties

- A VMD has a rational normal curve section and they have the same Betti table.

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Table 1 minimal degree varieties

- A VMD has a rational normal curve section and they have the same Betti table.

On the other hand, P. del Pezzo(1886) and E. Bertini(1907) classified all varieties of minimal degree:

- $X$ is of minimal degree $\Leftrightarrow$ $X$ is 2-regular ACM (characterization)

if and only if $X$ is (a cone of) one of the following (classification);

(a) a quadric hypersurface;

(b) a Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(c) a rational normal scroll, i.e. $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^{\Sigma a_i + d}$, where $\mathcal{E} \simeq \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_i \geq 1$.

Also See the paper "On Varieties of Minimal Degree (A centennial Account)-1987" due to D. Eisenbud and J. Harris.

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# 2-regular in a few steps

- Many geometric information on $X$ can be read off from the table (e.g. gonality, genus, degree bound and multisecant to $X$).
- $X$ satisfies $\mathbf{N}_{2,p}$ or $\mathbf{N}_p$ (i.e., 2-regular until $p$-th step) if $\beta_{i,j}(X) = 0, 1 \leq i \leq p, j \geq 2$.

| | 0 | 1 | 2 | 3 | $\cdots$ | $p$ | $p+1$ | $\cdots$ | $\triangle$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | – | $\cdots$ | – | | $\cdots$ | – |
| 1 | – | $\beta_{1,1}$ | $\beta_{2,1}$ | $\beta_{3,1}$ | $\cdots$ | $\beta_{p,1}$ | $\beta_{p+1,1}$ | $\cdots$ | $\beta_{\triangle,1}$ |
| 2 | – | – | – | – | $\cdots$ | – | $\beta_{p+1,2}$ | $\cdots$ | $\beta_{\triangle,2}$ |
| $\vdots$ | – | – | $\cdots$ | – | $\ddots$ | $\cdots$ | $\vdots$ | $\vdots$ | $\cdots$ |

- For a variety $X$ satisfying $\mathbf{N}_{2,p}$, if $X \cap \Lambda$ is finite for a linear space $\Lambda$ of dimension $p$, then they are linearly independent as a scheme. (Eisenbud-Green-Hulek-Popescu 2005)

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# The Betti table of smooth curves

Let $C$ be a smooth curve of genus $g$ and gonality $\mathrm{gon}(C)$ completely embedded in $\mathbb{P}^r$ by $\mathcal{L}$. Suppose $\deg(\mathcal{L}) = 2g+1+p \geq 4g-3$ and $r = h^0(C,\mathcal{L}) - 1 = g+p+1$. Then we have the following:

- $\beta_{i,1}(C) \neq 0 \Longleftrightarrow 1 \leq i \leq r - \mathrm{gon}(C)$ (Ein-Lazarsfeld);
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| | 0 | 1 | $\cdots$ | $p$ | $p+1$ | $\cdots$ | $r-\mathrm{gon}(C)$ | $\cdots$ | $r-1$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | $-$ | $\cdots$ | $-$ | $\cdots$ | $-$ | $-$ | $-$ | $-$ |
| 1 | $-$ | $\beta_{1,1}$ | $\cdots$ | $\beta_{p,1}$ | $\beta_{p+1,1}$ | $\cdots$ | $\beta_{r-\mathrm{gon}(C),1}$ | $-$ | $-$ |
| 2 | $-$ | $-$ | $\cdots$ | $-$ | $\beta_{p+1,2} \neq 0$ | $\cdots$ | $\cdots$ | | $\beta_{r-1,2} = g$ |

- In addition, there exists $(p+3)$-secant $(p+1)$-plane by the geometric Riemann-Roch and $\beta_{p+1,2} \neq 0$. So $\mathbf{N}_{2,p+1}$ does not hold for $X$.

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| 1 | – | $\beta_{1,1}$ | $\cdots$ | $\beta_{p,1}$ | $\beta_{p+1,1}$ | $\cdots$ | $\beta_{r-\mathrm{gon}(C),1}$ | – | – |
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- **$d$-regular in a few steps for $d \geq 2$.**

# $d$-regular in a few steps for $d \geq 2$ in general

- More generally, one says that $X$ satisfies $\mathbf{N}_{d,p}$ for $d \geq 2$ if $\beta_{i,j}(X) = 0, 1 \leq i \leq p, \;\; j \geq d$, i.e. $X$ is $d$-regular until $p$-th step.
  Note that $X$ is $d$-regular if $\mathbf{N}_{d,p}$ holds for all $p \geq 1$.

| | 0 | 1 | 2 | $\cdots$ | $p$ | $p+1$ | $\cdots$ | $\triangle$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | — | — | $\cdots$ | — | — | $\cdots$ | — |
| 1 | — | $\beta_{1,1}$ | $\beta_{2,1}$ | $\cdots$ | $\beta_{p,1}$ | $\beta_{p+1,1}$ | $\cdots$ | $\beta_{\triangle,1}$ |
| 2 | — | $\beta_{1,2}$ | $\beta_{2,2}$ | $\cdots$ | $\beta_{p,2}$ | $\beta_{p+1,2}$ | $\cdots$ | $\beta_{\triangle,2}$ |
| $\vdots$ | — | — | $\cdots$ | $\cdots$ | $\cdots$ | $\vdots$ | $\cdots$ | $\ddots$ |
| $d-1$ | — | $\beta_{1,d-1}$ | $\beta_{2,d-1}$ | $\cdots$ | $\beta_{p,d-1}$ | $\beta_{p+1,d-1}$ | $\cdots$ | $\beta_{\triangle,d-1}$ |
| $d$ | — | — | — | $\cdots$ | — | $\beta_{p+1,d}$ | $\cdots$ | $\beta_{\triangle,d}$ |
| $\vdots$ | | $\cdots$ | — | $\ddots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\ddots$ |
| $\square$ | — | — | — | — | — | $\beta_{p+1,\square}$ | $\cdots$ | $\beta_{\triangle,\square}$ |

- We have also some geometric properties for property $\mathbf{N}_{d,p}$.

# $d$-regular in a few steps for $d \geq 2$ in general

- More generally, one says that $X$ satisfies $\mathbf{N}_{d,p}$ for $d \geq 2$ if $\beta_{i,j}(X) = 0, 1 \leq i \leq p, \ \ j \geq d$, i.e. $X$ is $d$-regular until $p$-th step. Note that $X$ is $d$-regular if $\mathbf{N}_{d,p}$ holds for all $p \geq 1$.

| | 0 | 1 | 2 | $\cdots$ | $p$ | $p+1$ | $\cdots$ | $\triangle$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | $\cdots$ | – | – | $\cdots$ | – |
| 1 | – | $\beta_{1,1}$ | $\beta_{2,1}$ | $\cdots$ | $\beta_{p,1}$ | $\beta_{p+1,1}$ | $\cdots$ | $\beta_{\triangle,1}$ |
| 2 | – | $\beta_{1,2}$ | $\beta_{2,2}$ | $\cdots$ | $\beta_{p,2}$ | $\beta_{p+1,2}$ | $\cdots$ | $\beta_{\triangle,2}$ |
| $\vdots$ | – | – | $\cdots$ | $\cdots$ | $\cdots$ | $\vdots$ | $\cdots$ | $\ddots$ |
| $d-1$ | – | $\beta_{1,d-1}$ | $\beta_{2,d-1}$ | $\cdots$ | $\beta_{p,d-1}$ | $\beta_{p+1,d-1}$ | $\cdots$ | $\beta_{\triangle,d-1}$ |
| $d$ | – | – | – | $\cdots$ | – | $\beta_{p+1,d}$ | $\cdots$ | $\beta_{\triangle,d}$ |
| $\vdots$ | | $\cdots$ | – | $\ddots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\ddots$ |
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|---|---|---|---|---|---|---|---|---|
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# Degree upper bound for $\mathbf{N}_{d,e}$

**Theorem** [Ahn-Han-K, preprint]
Suppose $X \subset \mathbb{P}^{n+e}$ satisfies $\mathbf{N}_{d,e}$, $d \geq 2$.

- $\deg(X) \leq \binom{d-1+e}{e}$;
- $\deg(X) = \binom{d-1+e}{e}$ if and only if $X$ is ACM with $d$-linear resolution.

**Remark**

- $\mathbf{N}_{2,e}$ iff $\deg(X) = e+1$ iff $X$ is 2-regular ACM iff $X$ is a VMD.
- Eisenbud-Green-Hulek-Popescu call this property (for $d = 2$) "the syzygetic rigidity" in 'Restricting linear syzygies'(2005).
- **[Problem]** Does $\mathbf{N}_{d,e}$ property imply the $d$-regularity of $X$ for $d \geq 3$? This is true for $d = 2$.

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# Basic ideas

## How to prove the above Theorem?

- Take the picture explaining the degree of $X$ and
- Use the graded ellimination mapping cone sequence to control the minimal free resolution of $R/I_X$ as an $S_e$-module ;
- Interprete it locally by sheafification!
- Finally, $X$ is $d$-linear ACM if and only if $R/I_X$ is a free graded $S_e$-module, isomorphic to

$$\bigoplus_{0 \le i \le d-1} S_e(-i)^{\binom{e-1+i}{i}} \simeq R/I_X,$$

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- **Elimination mapping cone sequence**

Let $S_1 = k[x_1, \ldots, x_{n+e}] \subset R = k[x_0, x_1 \ldots, x_{n+e}]$
Let $M$ be a graded $R$-module (so, $M$ is also a graded $S_1$-module).
Then, we have a natural long exact sequence:

$$\mathrm{Tor}_i^R(M)_{i+j} \to \mathrm{Tor}_{i-1}^{S_1}(M)_{i-1+j} \xrightarrow{\times x_0} \mathrm{Tor}_{i-1}^{S_1}(M)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(M)_{i-1+j+1}$$

whose connecting homomorphism is induced by the multiplication map $\times x_0 : M(-1) \to M$.

- **Elimination mapping cone sequence** gives us the following:

  - $\mathrm{depth}_R(R/I_X) = \mathrm{depth}_{S_1}(R/I_X)$;
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$X \subset \mathbb{P}^{n+e}$ is ACM if $\mathrm{depth}\ R/I_X = \dim R/I_X = \dim(X) + 1$.

In particular, ACM varieties with $d$-linear resolution are very special:

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# Multisecant spaces for property $\mathbf{N}_{d,p}$, $1 \leq p \leq e$

Suppose $X \subset \mathbb{P}^{n+e}$ satisfies $\mathbf{N}_{d,p}$, $1 \leq p \leq e$. Then,

- $\mathbf{N}_{d,1}$ means $X$ is cut out by equations of degree at most $d$. So, there is no $(d+1)$-secant line to $X$ and furthermore,
- A $d$-secant line to $X$ is contained in any defining equation of degree $\leq d-1$.

We have the following generalization:

**Proposition (the generalized Bezout's Theorem)** [Ahn-Han-K]

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**Proposition** [Ahn-Han-K, preprint]
Consider the $d$-secant locus $\Sigma_{q,d}(X)$ through $q \in \mathrm{Sec}(X) \setminus X$. Then, $\Sigma_{q,d}(X) := \{x \in X \mid {\pi_q}^{-1}(\pi_q(x))$ has length d $\}$. Property $\mathbf{N}_{d,2}$, $d \geq 2$ implies the following:

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■ **The structure of the Betti Table.**
We are intereested in the firsr linear strand starting from quadric equations:

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**Elementary questions**: $X^n \subset \mathbb{P}^{n+e}$: nondegenerate, irreducible and reduced defined over $K = \overline{K}$ of char$(K) \geq 0$.

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- The (next-to-simplest) Betti table of a del Pezzo variety with

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| | 0 | 1 | 2 | 3 | $\cdots$ | $i$ | $\cdots$ | $e-1$ | $e$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | – | $\cdots$ | – | $\cdots$ | – | – |
| 1 | – | $\beta_{1,1}$ | $\beta_{2,1}$ | $\beta_{3,1}$ | $\cdots$ | $\beta_{i,1}$ | $\cdots$ | $\beta_{e-1,1}$ | – |
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# Upper bound on quadrics and higher linear syzygies

- **Castelnuovo's simple proof.**

$\Gamma = X \cap \mathbb{P}^e$ is a set of $d$-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

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**Theorem** [Basic Inequality] (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1$.
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$\Gamma = X \cap \mathbb{P}^e$ is a set of $d$-points in general position for general $\mathbb{P}^e$.
Since $d \geq e+1$, take a subset $\Gamma' = \{p_1, p_1, \ldots, p_{e+1}\} \subset \Gamma \subset \mathbb{P}^e$.
$h^0(\mathcal{I}_X(2)) \leq h^0(\mathcal{I}_\Gamma(2)) \leq h^0(\mathcal{I}_{\Gamma'}(2)) = \binom{e+2}{2} - (e+1) = \binom{e+1}{2}$.

- Inner projection method is more powerful.

**Theorem** [Basic Inequality] (Han-K, 2015)

- $\beta_{i,1}(X) \leq \beta_{i,1}(X_q) + \beta_{i-1,1}(X_q) + \binom{e}{i}, \quad i \geq 1.$
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- [Green, 1984] If $\beta_{p,1} \neq 0$, then $h^0(\mathcal{I}_X(2)) \geq \binom{p+1}{2} = \frac{(p+1)p}{2}$;
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- [Han-K, 2015] Using the above basic inequality under inner projection, we have the following: $X \subset \mathbb{P}^{n+e}$ : irreducible, reduced (not necessarily smooth).

$$\beta_{i,1}(X) \leq i\binom{e+1}{i+1}, \quad i \geq 1$$

- Furthermore, $\beta_{i,1}(X) = i\binom{e+1}{i+1}$ for some $1 \leq i \leq e$ if and only if $X$ is a VMD iff $X$ is a 2-regular ACM variety.

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We also characterize Fano varieties as follows:

**Theorem** [Han-K, 2015]
Unless $X$ is a variety of minimal degree, then we have

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and in particular,

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# The first strand of the Betti table in the cubic world

■ **The structure of Betti Tables.**
We are intereested in the firsr linear strand starting from cubic equations assuming that $X$ has no quadrics:

$$\beta_{1,2}(X), \beta_{2,2}(X), \ldots, \beta_{e,2}(X), \ldots.$$

■ **We have many parallel problems as in the quadratic world.**

- How many cubics are required for $\beta_{p,2}(X) \neq 0$ for $1 \leq p \leq e$? ;
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# Notion of $\mathrm{ND}(2)$

An irreducible variety $X^n \subset \mathbb{P}^{n+e}$ is called a $\mathrm{ND}(m)$(i.e., nondegenerate in degree $m$) variety if for a general $\Lambda$ of dimension $e$, $(I_{X\cap\Lambda})_m = 0$.

- $\mathrm{ND}(1) \Leftrightarrow$ nondegenerate $\Leftrightarrow$ not contained in any hyperplane.
- $\mathrm{ND}(2) \Leftrightarrow$ nondegenerate in degree 2 $\Leftrightarrow$ not contained in any quadric after linear sections.
- The Betti table of $\mathrm{ND}(d-1)$ variety $X$ in $\mathbb{P}^{n+e}$:

| | 0 | 1 | 2 | 3 | $\cdots$ | $i$ | $i+1$ | $\cdots$ | $\triangle$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | – | – | – | $\cdots$ | – | – | $\cdots$ | – |
| 1 | – | – | – | – | $\cdots$ | – | – | $\cdots$ | – |
| $\vdots$ | – | – | $\cdots$ | – | $\ddots$ | $\cdots$ | $\vdots$ | $\vdots$ | $\cdots$ |
| $d-1$ | – | $\beta_{1,d-1}$ | $\beta_{2,d-1}$ | $\beta_{3,d-1}$ | $\cdots$ | $\beta_{i,d-1}$ | $\beta_{i+1,d-1}$ | $\cdots$ | $\beta_{\triangle,d-1}$ |
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# Upper bound of $\beta_{p,2}(X)$ and $K_{p,2}$ Theorem

Theorem (Ahn, Han and K-, preprint)

*Suppose that $X^n \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(2)$ subscheme, defined over $K = \overline{K}$ of char $(K) = 0$. Then,*

- $\binom{e+2}{2} \leq \deg(X)$ *and* $h^0(\mathcal{I}_X(3)) \leq \binom{e+2}{3}$ .
- *In general,* $\beta_{p,2}(X) \leq \binom{p+1}{2}\binom{e+2}{p+2}$ *for* $p \geq 1$.
- *For the extremal cases, the following are equivalent:*
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  - (d) $I_X$ *has ACM* 3*-linear resolution.*

This also gives a natural $K_{p,2}$ theorem generalizing $K_{p,1}$-theorem because $\beta_{p,2}(X) = 0$ for $p > e$.

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# Degree lower bound and the upper bound on syzygies

**Theorem** [Ahn-Han-K] More generally, if $X \subset \mathbb{P}^{n+e}$ is a $\mathrm{ND}(d-1)$ variety defined over $K$ of char $(K) = 0$ then,

- $\binom{e+d-1}{d-1} \leq \deg(X)$ and $h^0(\mathcal{I}_X(d)) \leq \binom{e+d-1}{d}$ ;
- $\beta_{i,d-1}(X) \leq \binom{i+d-2}{d-1}\binom{e+d-1}{i+d-1}$ for $i \geq 1$. In particular, $\beta_{i,d-1}(X) = 0$ for $i \gneq e$.
- For the extremal cases, the following are equivalent:
  - $\deg(X) = \binom{e+d-1}{d-1}$ ;
  - $h^0(\mathcal{I}_X(d)) = \binom{e+d-1}{d}$ ;
  - $\beta_{i,d-1}(X)$ attains the maximum for some $1 \leq i \leq e$ ;
  - $X$ is ACM with $d$-linear resolution.

**Remark**

- For $d \geq 2$, $\beta_{i,d-1}(X) = 0$ for $i > e$ is a generalization of Green's $K_{p,1}$-Theorem ($d = 2$).

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## Remark

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# How to Prove? in case char$(K) = 0$

Let $m = (x_0, x_1, \ldots, x_{e-1})$ be the irrelevant maximal ideal in $\Lambda = \mathbb{P}^{e-1}$.
The graded Betti number of a $\mathrm{ND}(d-1)$-variety is less than or equal to that of the $m^d$, i.e.

$$\beta_{p,d-1}(X) \leq \beta_{p,d-1}(R/m^d) = \binom{e+d-1}{p+d-1}\binom{p+d-2}{d-1} \text{ for } p \geq 1$$

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(a) **Generic initial ideal** Consider $\mathrm{in}_\tau(g(I))$. For a general change $g$, $\mathrm{in}_\tau(g(I))$ is constant. We will call this the *generic initial ideal of I* w.r.t $\tau$ $Gin_\tau(I)$

(b) Other ingredients

- Use degree reverse lexicographic order;
- Cancellation principle;
- Eliahou-Kervaire theorem;
- $Gin(\bar{I}) = \frac{(Gin(I), x_n)}{(x_n)} = Gin(I)|_{x_n \to 0}$;
- $Gin(\bar{I}^{\mathrm{sat}}) = \bigcup_{k=0}^{\infty}(Gin(\bar{I}) : x_{n-1}^k) = (Gin(I)|_{x_n \to 0})|_{x_{n-1} \to 1}$.

(c) **Strategy** Find maximal possible Borel fixed set which contains $Gin(I_X)_3$!

- For all $\mathrm{ND}(2)$-varieties, $\deg(X) \geq \binom{e+2}{2}$ and 3-linear ACM varieties are called 'minimal degree varieties of the second kind'.
- **Problem**: What is a geometric properties /or classification of 'minimal degree varieties of the second kind'?

**Examples of varieties having ACM 3-linear resolution]**

- 3-minors of $4 \times 4$ generic symmetric matrix (i.e. $\mathrm{Sec}(v_2(\mathbb{P}^3)) \subset \mathbb{P}^9$);
- 3-minors of $3 \times (e+2)$ sufficiently generic matrices (e.g. $\mathrm{Sec}(RNS)$);
- $\mathrm{Sec}(v_3(\mathbb{P}^2))$;
- $\mathrm{Sec}(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1)$;
- Non-trivial 3-linear ACM smooth varieties are interesting! (L. Ein)
- Are they all the secant varieties of varieties of small degree? (M. Mella.)

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