Complete intersection constructions of toric degenerations of Fano varieties

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Overview

Goal

In this talk I will to describe a construction which takes as input a reflexive polytope P (or anti-canonically polarised toric Fano variety X_P) and returns a collection of pairs $(\tilde{P}, \phi)$ consisting of a polytope $\tilde{P}$ and a map $\phi\colon \text{Ambient}(\tilde{P}) \to \mathbf{Z}^r$ which determines a toric complete intersection of $X_{\tilde{P}}$ isomorphic to X_P.

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Applications

- Generate possibly previously unknown Fano 4-folds from reflexive polytopes by studying general elements of the linear system defining X_P in $X_{\tilde{P}}$.
- Use Mirror Symmetry techniques to identify which of these Fano manifolds appear on existing lists.
- Provide a testing ground for extending results on Mirror Symmetry for (orbifold) del Pezzo surfaces to higher dimensions.

History and references I

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Four-dimensional Fano complete intersections

Here we searched all possible four-dimensional Fano complete intersections X, a section of $L = \oplus_i L_i$ in Y such that:

- Y is smooth.
- Y is Fano ($-K_Y$ is ample).
- The line bundles L_i are nef.
- $-(K_Y + \sum_i L_i)$ is ample (so X is Fano by adjunction).

We classified such varieties X by their *Quantum period sequence*, and found 527 Fano manifolds not appearing on a previous list.

History and references II

The process for computing the Quantum period sequence uses Mirror Symmetry to replace the Fano manifold with a mirror-dual *Laurent polynomial*, via a technique going back to Givental/Hori–Vafa. We asked if there was a systematic way of *inverting* this process. That is, can we recover a Fano manifold from a candidate mirror?

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Laurent Inversion

In this article we describe such a technique, which we call *Laurent Inversion*, and demonstrate its use by exhibiting a Fano manifold not in the previous 527 or other list. This talk is essentially the sum of my recent efforts to better understand this technique.

Dual perspectives on polytopes

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- As the convex hull of its vertices (or integral points): $P = \mathrm{conv}(v | v \in \mathrm{Verts}(P))$.
- As the intersection of a (possibly redundant) collection of inequalities: $P = \cap_i \{m | \langle u_i, m \rangle \geq a_i\}$.

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Remark

Writing N_Q for $N \otimes_{\mathbf{Z}} \mathbf{Q}$, given a reflexive polytope $P \subset N_{\mathbf{Q}}$ we define $Q := P^\circ$ to be its polar polytope in $M := \hom(N, \mathbf{Z})$.

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Conventions on varieties

For (Fano) $P \subset N_{\mathbf{Q}}$ the toric variety X_P is defined by its *spanning fan*, for $Q \subset M_{\mathbf{Q}}$ the toric variety X_Q is defined by its *normal fan*.

Scaffolding: the N-side

The main idea of the construction is simple: rather than writing X_P as a convex hull of points, we take the convex hull of a collection of polytopes. This collection forms additional data which allows us to build an ambient space for X_P.

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Input data

Fix a pair $(P, \mathcal{T})$ where $P \subset N_{\mathbf{Q}}$ is a reflexive polytope and $\mathcal{T}$ is a tuple of integral polytopes $T \in \mathcal{T}$ we will refer to $\mathcal{T}$ as the *type*.

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Definition

A *scaffolding* $\mathcal{S}$ is a finite collection of elements $S = (v, -a) \in \tilde{N} := N \oplus \mathbf{Z}^{\mathcal{T}}$ such that

$$\mathrm{conv}(v + \sum_{T \in \mathcal{T}} a_T T | S \in \mathcal{S}) = P$$

where a_T is the T component of $a \in \mathbf{Z}^{\mathcal{T}}$.

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The construction works for any type polytopes, but I will generally assume that $\mathrm{Verts}(T) = T \cap N$ *for all* T*. It is reasonable to restrict to the case* T *is a simplex.*

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We introduce some terminology for scaffoldings.

- A *strut* is an element $S \in \mathcal{S}$.
- The *rank* of $\mathcal{S}$ is the number of struts.
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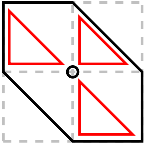
Rays of the ambient space

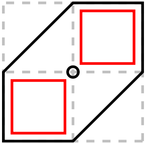
Define $\tilde{\Sigma}_{P,\mathcal{S}}(1)$ to be the collection of rays in $\tilde{N}_{\mathbf{Q}}$ generated by the integral points

$$\mathcal{S} \cup \{(v, e_T) | v \in \mathrm{Verts}(T)\}$$

where e_T is the indicator of T in $\mathbf{Z}^{\mathcal{T}}$.

Examples of scaffoldings





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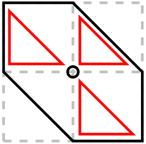
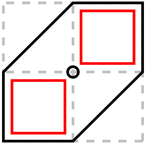


The rays given by our construction fit into the matrix

$$\rho := \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}$$



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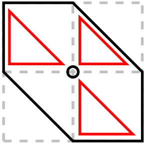

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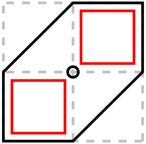

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These examples will recover classical constructions of a degree 6 del Pezzo surface. In these examples we also see the ray matrix is (nearly) in echleon form, a property we see again later.

Scaffolding: the M-side

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Defining $\tilde{Q}$

Recall the set of integral points we defined in $\tilde{N}$

$$R := \mathcal{S} \cup \{(v, e_T) | v \in \mathrm{Verts}(T)\}$$

Define a function $\phi\colon R \to \mathbf{Z}$ by setting

$$\phi(r) = \begin{cases} -1 & r \in \mathcal{S} \\ 0 & r \notin \mathcal{S} \end{cases}$$

Let $\tilde{Q} := \{(u, b) \in \tilde{M} : \langle r, (u, b) \rangle \geq \phi(r)\}$

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Define $Y := X_{\tilde{Q}}$ via the normal fan $\tilde{\Sigma}$ of $\tilde{Q}$. There are divisors D_T defined for each $T \in \mathcal{T}$.

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The linear system

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$$D_T := \sum_{v \in \text{Verts } T} \langle (v, e_T) \rangle$$

Observe that each standard basis vector $e_T^\star$ in $\tilde{M}$ is a functional on $\tilde{N}$ evaluating to 1 on each ray of $\{(v, e_T) | v \in \text{Verts}(T)\}$ and non-positively on all other rays.

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This definition of Y ensures that $-K_Y - \sum_T D_T$ is an ample divisor. For this to be a useful definition however there are a number of things to check.

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The toric complete intersection defined by the collection $\{e_T^\star : T \in \mathcal{T}\}$ is isomorphic to X_P.

We will omit all the details of the proofs, but the proofs follow from a reinterpretation of the scaffolding on the M-side, and we will explain this idea.

PL functions and M-side struts

Let Q be the polar polytope of P, recall this is integral if P is reflexive. For each $T \in \mathcal{T}$ there is a decomposition $\mathcal{Q}_T$ of Q given by the normal fan of T.

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The PL map ι

Define the map

$$\iota(u) = \left(u, \left(T \mapsto -\min_{v \in \mathrm{Verts}(T)} \langle u, v \rangle\right)\right) \in M \oplus \mathbf{Z}^{\mathcal{T}}$$

Restricting ι to Q we have a PL embedding of Q into the boundary of $\tilde{Q}$.

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Supporting hyperplanes

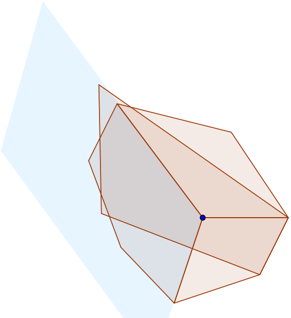
Given a facet F of Q, we say an affine hyperplane H in $\tilde{M}$ *supports* F if $\iota(F) \subset H \cap \iota(Q)$ and $H \cap \iota(\mathrm{int}\, Q) = \emptyset$.

PL functions and M-side struts

Our main result establishes the correspondence between the N and M side versions of the scaffolding.

Duality result

There is a bijection between affine hyperplanes H supporting F and struts of type $\mathcal{T}$ meeting $v \in \mathrm{Verts}\, P$, where F is the facet of Q dual to v.



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Point struts

Take *any* type $\mathcal{T}$ and form the trivial scaffolding $\mathcal{S}$ indexed by Verts P

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The ambient space is a non-compact toric variety in which the toric variety X_P degenerates to a collection of toric varieties, this is referred to as the *Mumford degeneration* by Gross–Siebert and forms an important ingredient in their procedure.

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Single strut

At the other extreme take $\mathcal{T}$ to be the singleton set consisting of P. Then there is a scaffolding $\mathcal{S}$ consisting of the single element $(0,-1) \in N \oplus \mathbf{Z}$. This ambient space is simply the *projective cone* over X_P.

Designing algorithms to find Fano 4-folds

There are 473,800,776 four dimensional reflexive polytopes. Ideally we would design a comprehensive search of this list for all types of scaffolding and recover a wide class of Fano 4-folds, and a good lower bound on the total number of Fano 4-folds. However, there are obvious computational difficulties.

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In this section we briefly illustrate some simple ways of identifying polytopes which certain scaffoldings are likely to apply to and how to check these ambient spaces.

Techniques for finding line segment scaffoldings

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For each vertex v of $Q := P^\circ$ say v is smooth if the tangent cone at v is. Suppose that the codimension of the linear space spanned by vertices which are not smooth is equal to one. Then there is a primitive dual vector $f \in N$ and we use the line segment $[0, f]$ to scaffold P.

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Lines of nodes

Suppose there is a 3-cone in the spanning fan of P which is a conifold, that is, the cone over a square at height one. Geometrically there is a $\mathbf{C}^{\star}$ of ODPs in the four-fold X_P. Suppose all singularities of X_P lie in the closure of this $\mathbf{C}^{\star}$. Scaffolding this with horizontal or vertical line segments produces an ambient space which may smooth this line of nodes.

Finding smoothings

We do not list an algorithm for checking whether these ambient spaces define smoothings, but indicate its main features.

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Quasismoothness

Recall that the toric variety $Y = X_{\tilde{Q}}$ is a GIT quotient of a vector space V by a complex torus $\mathbb{T}$. We check smoothness of the complete intersection in $V \backslash Z$ where Z is the vanishing locus of the irrelevant ideal. Even if we pass this test we need to study the interesections with the singularities of the ambient.

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Avoiding singularities

We also check this in the cone V, restricting the equations to certain strata and checking if the complete intersection equations give zeroes which would give singularities.

Full-dimensional scaffoldings

At the other extreme from line segment scaffolding we can study types $\mathcal{T}$ such that each T is a standard simplex, the T lie in pairwise orthogonal subspaces and their dimensions sum to $\dim P = \dim N$. We call such scaffoldings *full*.

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For a scaffolding as above the ray matrix ρ is (after an easy change of variables) in echelon form. Since the kernel of the ray map is the GIT weight data we can read this information from the scaffolding.

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This was the original setting of this construction in CKP14. The ray matrix exactly encodes the scaffolding, and since the rank of the ambient space is the number of struts iterating over matrices encoding a small number of struts ought to produce interesting examples.

Reading off the weight matrix

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$$v_T := \sum_{e \in \mathcal{B}_T} \langle e^\star, v \rangle e, \text{ and } b_T := a_T + \sum_{e \in \mathcal{B}_T} \langle e^\star, v \rangle$$

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Weight matrix

Fix a strut $S = (v, a) \in \mathcal{S}$. This defines a row R_S of the weight matrix. R_S is the element of

$$\mathbf{Z}^{\mathcal{S}} \oplus \bigoplus_{T \in \mathcal{T}} \mathbf{Z}^{\dim T + 1} \cong \mathbf{Z}^{\mathcal{S} \cup \mathcal{B} \cup \mathcal{T}}$$

given by concatenating e_S with vectors $(-v_T, b_T)$ for each T.

The weight matrix, example

The weight matrix has the general form

$$D = \left[\begin{array}{c|c} I_{|\mathcal{S}|} & A \end{array} \right]$$

Where the columns of A are further divided into blocks of length $\dim T + 1$ for each $T \in \mathcal{T}$. As a daft example, we can consider precisely those Fano triangles covered by one dilated and translated standard simplex.

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Weight data

The ambient space is always $Y = \mathbf{P}(1, a, b, c)$ for some $a, b, c \in \mathbf{Z}_{>0}$. Well-formedness is equivalent to the primitivity of the vertices of the triangle. the linear system we consider is $L = \mathcal{O}(a + b + c)$ and thus we have that $-K_Y - L = \mathcal{O}(1)$.

Data and Examples

We'll divide outcomes into surface, threefold and fourfold cases. Starting with the ten del Pezzo surfaces we have the following easy result:

Proposition

All of the 10 del Pezzo surfaces can be recovered by scaffolding. Every reflexive polytope admits a scaffolding producing a general section which is one of the eight del Pezzo surfaces with very ample anti-canonical bundle.

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We clearly have no such result in 3 or 4 dimensions. In this section we review the known constructions for these dimensions as well as for log del Pezzo surfaces.

del Pezzo surfaces

Continuing our example from earlier we can verify that the scaffoldings on the hexagon produce

- $\mathbf{P}^2 \times \mathbf{P}^2$ as a section of $\mathcal{O}(1,1) \oplus \mathcal{O}(1,1)$ or,
- $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ as a section of $\mathcal{O}(1,1,1)$.

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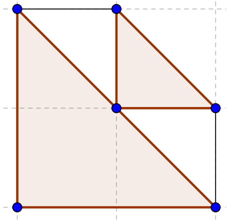

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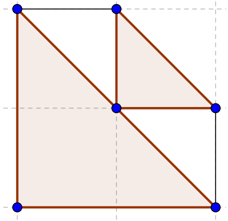


This scaffolding produces the ambient space $\mathbf{P}^2 \times \mathbf{P}^1$ with linear system $\mathcal{O}(2,1)$. A general section is a graph over $\mathbf{P}^2$ except at the vanishing of a pair of quadrics, at which the surface contains a line.

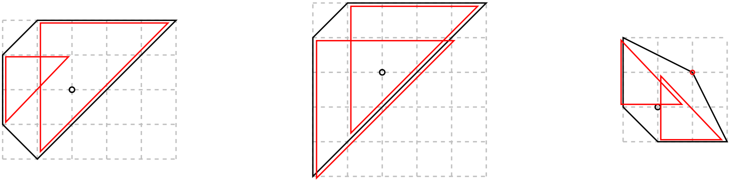
log del Pezzo surfaces

As part of the classification of del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities we used scaffoldings to construct a number of models, summarized in the following result.

Proposition

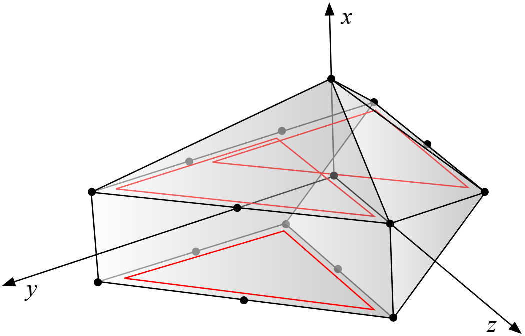
Of the 26 del Pezzo surfaces with at most $\frac{1}{3}(1,1)$ singularities which admit a toric degeneration exactly 24 can be recovered via a scaffolding construction.

For example, the log del Pezzo surface $X_{2,5/3}$ ($2\ \frac{1}{3}(1,1)$ singularities, degree $5/3$) has a rank two ambient space (left).



Threefolds

There are 98 Fano threefolds with very ample anti-canonical bundle. In *Quantum Periods for 3-Dimensional Fano Manifolds* Coates–Corti–Galkin–Kasprzyk computed the Quantum Period of all smooth Fano 3-folds. Of these, 86 are either smooth toric or have complete intersection constructions. All of these are examples of (possibly trivial) scaffoldings.



Threefold example

For example, consider the blowup of $\mathbf{P}^3$ in the intersection of two cubics. We can write this as a section of $\mathcal{O}(1,3)$ in $\mathbf{P}^1 \times \mathbf{P}^3$.

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$$D := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Where the order indicates the identity block. We can read off the struts as

- $v = (0,0,0), a = 1,$
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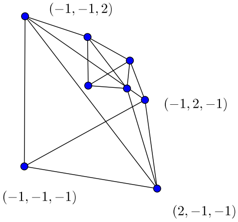
This produces a scaffolding by a pair of tetrahedra.



Fourfolds: CKP14

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Of course, for the smooth toric Fano 4-folds the trivial scaffolding suffices to reproduce the variety.

Fourfolds: generating new examples

Applying the algorithms mentioned above to the first 5000 polytopes in the Kreuzer–Skarke list we recover an additional 19 Fano smooth varieties which are different from the ones previously recovered. There are a number of ways we can find 4-folds not in the list 527 from CKP14.

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- Y is not $\mathbf{Q}$-factorial ambient. Alternatively, resolving Y, $-K_Y - \sum_i L_i$ becomes strictly nef.
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Lines of nodes

We find 936 polytopes in the first 5000 of the reflexive 4-topes whose singular set is a line of ODPs. Of these the scaffolding construction produces a smoothing in 162 of these cases. Of these 15 are 'new'.

Laurent Polynomials

Mirror Symmetry appears in a central, though rather superficial way in our construction. In particular, from a given scaffolding $\mathcal{S}$ on P one can construct a Laurent polynomial $f_{\mathcal{S}}$, whose Newton polytope is precisely P.

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Laurent polynomial

For each strut $S \in \mathcal{S}$, define the strut polynomial

$$f_S := z^v \prod_{T \in \mathcal{T}} \left(\sum_{m \in T} z^m \right)^{a_T}$$

we then define a Laurent polynomial $f_{\mathcal{S}}$ by summing over these,

$$f_{\mathcal{S}} := \sum_{S \in \mathcal{S}} f_S$$

By construction $\mathrm{Newt}(f_{\mathcal{S}}) = P$.

Laurent Polynomials

In fact, this is the mirror-dual to X as a complete intersection given by Givental/Hori–Vafa, indeed, by construction we know that.

Proposition

The Hori–Vafa construction applied to the toric variety $Y := X_{\tilde{P}}$ *and linear system* L *outputs an affine toric variety* X' *and superpotential* $W : X' \to \mathbf{C}$*. There is a birational map* ϕ *from* $(\mathbf{C}^\star)^n$ *to* X' *such that* $\phi^\star W = f_{\mathcal{S}}$*.*

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Of course this proposition only makes sense when a Mirror Theorem can be applied, so Y should be an orbifold and care should be taken when $-K_Y - \sum_i L_i$ is not ample.

The End