The ideal of orthogonal representations of a graph

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joint work with J. Herzog, S. Saeedi Madani, A. Macchia

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Orthogonal representations of a graph

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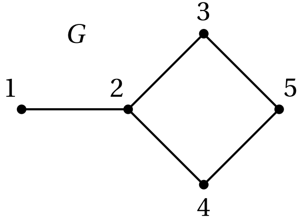

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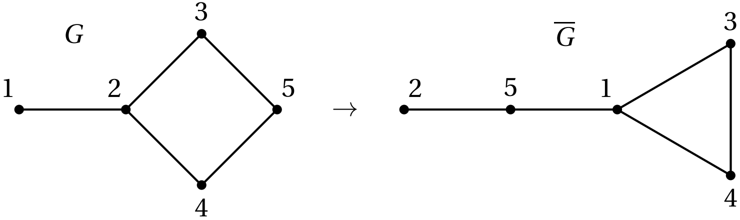

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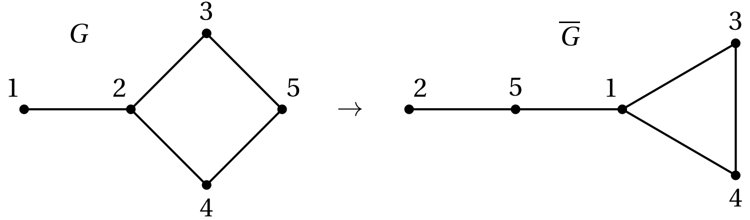

A map $\varphi : [n] \to \mathbb{R}^d$, $i \mapsto (x_{i1}, \ldots, x_{id})$ defines an *orthogonal representation* of G if for every $\{i, j\} \in E(\overline{G})$,

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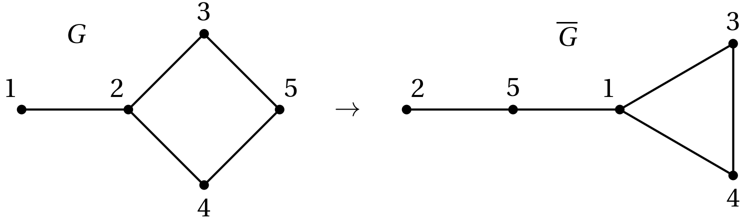

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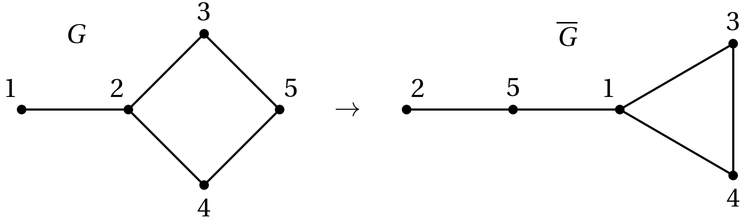

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- Intimately related to important combinatorial properties of graphs.

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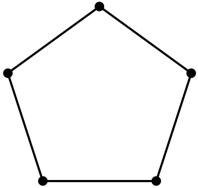

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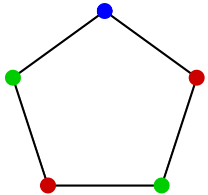

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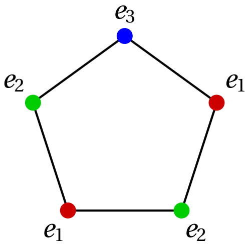

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- Take vertex coloring of $\overline{G}$ with $\chi(\overline{G})$ colors.
- Associate to vertices with color i the vector $e_i \in \mathbb{R}^{\chi(\overline{G})}$.

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- Adjacency means that the letters can be confused.
- $$\begin{array}{c}\text{Maximum number of 1-letter messages} \\ \text{which cannot be confused pairwise}\end{array} = \alpha(G)$$
- $\alpha(G)$ = the maximum cardinality of independent set.

Motivation

- $G^k = (V^k, E(k))$ the k^{th} power of $G = (V, E)$:
  Two different vertices $(v_1, \dots, v_k), (w_1, \dots, w_k) \in V^k$ are connected by an edge in $E(k)$ if $\{v_i, w_i\} \in E$ whenver $v_i \neq w_i$.

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- $\alpha(G) \leq \Theta(G)$.
- Computational complexity of $\Theta(G)$ unknown, and the value of $\Theta(G)$ is unknown for small graphs such as C_7.

The theta function

The *theta function* of G (Lovász, 1979) is

$$\vartheta(G) = \min_{(u_i),c} \max_{i\in[n]} \frac{1}{(c^T u_i)^2},$$

where the minimum is taken over all ortho**normal** representations $(u_i : i \in V)$ of G in $\mathbb{R}^d$, all unit vectors $c \in \mathbb{R}^d$ and integers $d \geq 1$.

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- The theta function can be computed in terms of $\overline{G}$ as

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Theorem (Lovász, 1986) *For every graph G,*

$$\alpha(G) = \omega(\overline{G}) \leq \vartheta(G) \leq \chi(\overline{G}),$$

where $\omega(\overline{G})$ is the size of the largest clique in $\overline{G}$ and $\chi(\overline{G})$ is the vertex-chromatic number of $\overline{G}$.

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- In general, the above inequalities are strict. If $\omega(G) = \chi(G)$, the graph G is called *perfect*. For example, this is the case for chordal graphs and for bipartite graphs and their complements.

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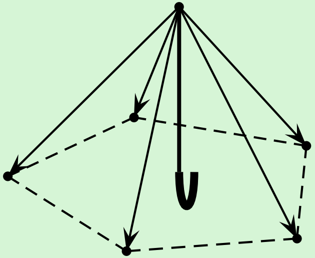

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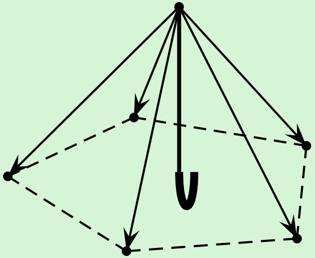

From Combinatorics to Commutative Algebra

From an algebraic point of view, the set of all orthogonal representations of a graph G is the vanishing set in $\mathbb{R}^{n\times d}$ of the ideal

$$L_{\overline{G}} = (x_{i1}x_{j1} + \cdots + x_{id}x_{jd} : \{i, j\} \in E(\overline{G}))$$

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as an ideal in the polynomial ring $T = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

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$$L_{\overline{G}} = (x_i x_j + y_i y_j : \{i, j\} \in E(\overline{G}))$$

  as an ideal in the polynomial ring $T = K[x_1, \dots, x_n, y_1, \dots, y_n]$.
- Binomial ideals well studied and we can use some of their theory.

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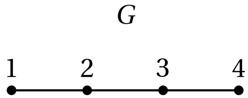

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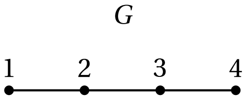

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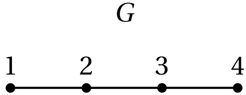

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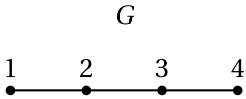

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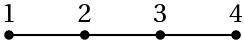

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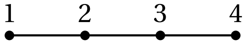

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- The claim follows. $\square$

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- True in the limit if $d \to \infty$. (ongoing work with Aldo Conca)

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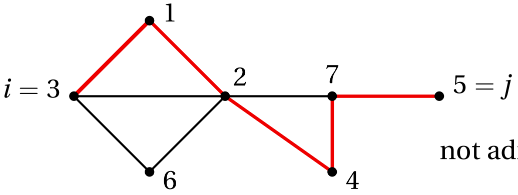

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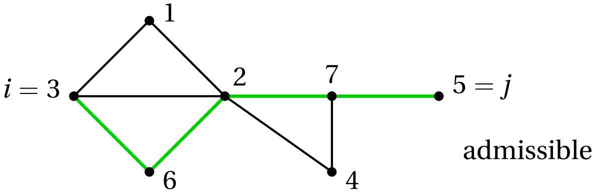

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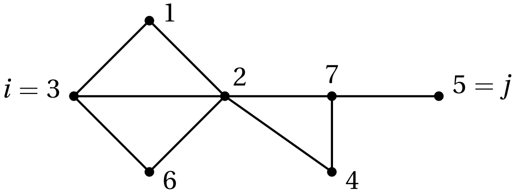

If π_{ij} is admissible, we attach to it the monomial

$$u_{\pi_{ij}} = \prod_{i_k > j} x_{i_k} \prod_{i_k < i} y_{i_k}.$$

Theorem (HMMW) *Let G be a graph on $[n]$ and assume that* $\mathrm{char}(K) \neq 2$. *Then, with respect to the lexicographic order on $T = K[x_i, y_i]$ induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$, the following elements form a Gröbner basis of the ideal Π_G:*

1. $u_{\pi_{ij}} b_{ij}$, *where π_{ij} is an odd admissible path and $b_{ij} = x_i y_j + x_j y_i$,*
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4. $$\begin{cases} y_b \prod_{h \in W} x_h & \text{if } b < h \text{ for every } h \in W \\ x_b \prod_{h \in W} y_h & \text{if } b > h \text{ for every } h \in W \end{cases},$$
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The proof is an application of *Buchberger's Algorithm*.

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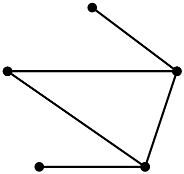

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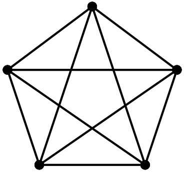

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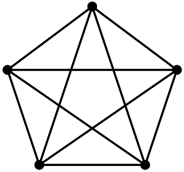

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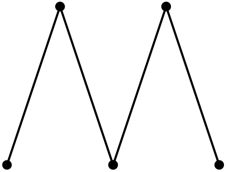
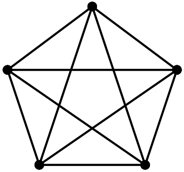


H

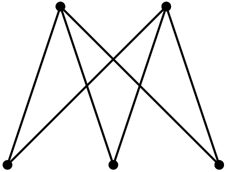
Primary decomposition of L_G for $\sqrt{-1} \notin K$

From now on we assume $\sqrt{-1} \notin K$. This is the case for $K = \mathbb{R}$.
Let H be an arbitrary connected graph on $[n]$.

- If H is not bipartite, then we denote by $\widetilde{H}$ the complete graph K_n on $[n]$.
- If H is bipartite on $V_1 \cup V_2$, $|V_1| = m$, $|V_2| = n - m$, then we denote by $\widetilde{H}$ the complete bipartite graph $K_{m,n-m}$ on $[n]$ with respect to the same bipartition.



K_5



$K_{2,3}$

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$$I_{\widetilde{G}_k} = \begin{cases} \left(x_i x_j + y_i y_j, x_i y_j - x_j y_i, x_h^2 + y_h^2 : \substack{1\le i<j\le \ell \\ 1\le h\le \ell}\right) & \text{if } \widetilde{G}_k = K_\ell \\ & \end{cases}$$

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Proposition (HMMW) *Let $\sqrt{-1} \notin K$. Then $Q_S(G)$ is a prime ideal for all $S \subset [n]$ and* $\operatorname{ht} Q_S(G) = |S| + n - b(S)$, *where $b(S)$ is the number of bipartite connected components of $G_{[n] \setminus S}$.*

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Theorem (HMMW) *Let G be a graph on $[n]$ and $\sqrt{-1} \notin K$. Then*

$$L_G = \bigcap_{S \subset [n]} Q_S(G)$$

is a (in general redundant) primary decomposition of L_G.

Corollary *Let G be a graph on $[n]$, and let $\sqrt{-1} \notin K$. Then*

$$\dim(T/L_G) = \max\{n - |S| + b(S) : S \subset [n]\}.$$

In particular, $\dim(T/L_G) \geq n + b$, *where b is the number of bipartite connected components of G. Moreover, if L_G is unmixed, then* $\dim(T/L_G) = n + b$.

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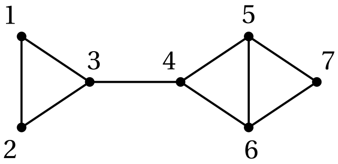

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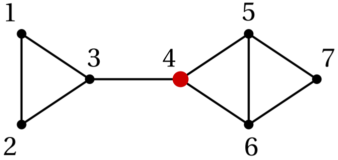

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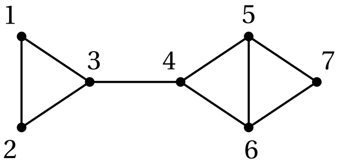

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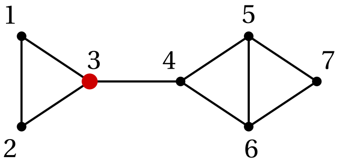

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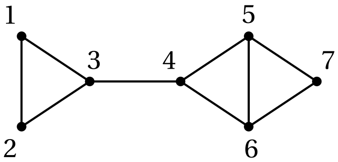

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Let $\mathcal{M}(G)$ be the set of all sets $S \subset [n]$ such that each $i \in S$ is either a cut point or a bipartition point of the graph $G_{([n]\setminus S)\cup\{i\}}$. In particular, $\varnothing \in \mathcal{M}(G)$.

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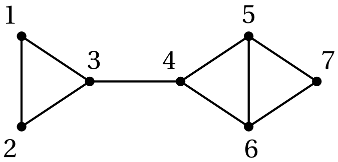

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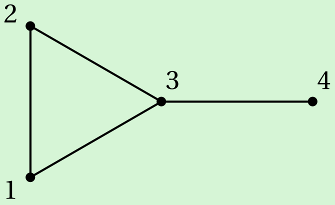
Theorem (HMMW) *Let G be a graph on $[n]$, $\sqrt{-1} \in K$ and $S \subset [n]$. Then $Q_S(G)$ is a minimal prime ideal of L_G if and only if $S \in \mathcal{M}(G)$.*

Example Let G be the graph



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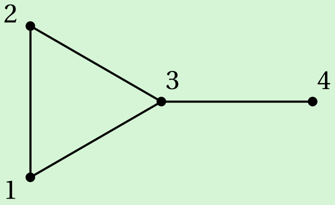

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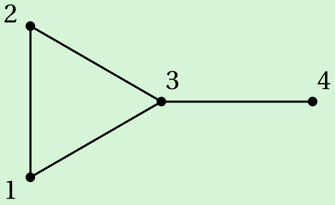

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$$L_G = Q_{\varnothing}(G) =$$

$$(x_1x_2 + y_1y_2, x_1x_3 + y_1y_3, x_1x_4 + y_1y_4, x_2x_3 + y_2y_3, x_2x_4 + y_2y_4, x_3x_4 + y_3y_4,$$
$$x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3,$$
$$x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2, x_4^2 + y_4^2)$$

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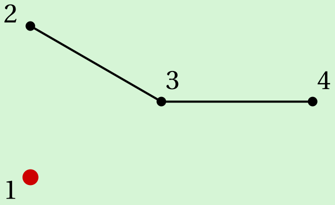

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$$L_G = Q_{\varnothing}(G) \cap Q_{\{1\}}(G) =$$

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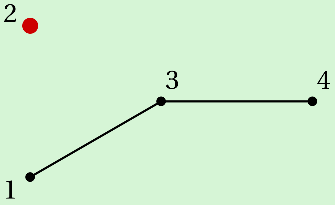

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$$L_G = Q_{\varnothing}(G) \cap Q_{\{1\}}(G) \cap Q_{\{2\}}(G) =$$

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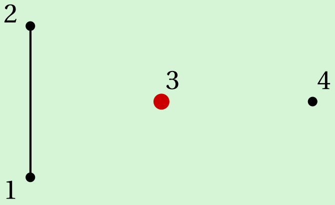

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$$L_G = Q_{\varnothing}(G) \cap Q_{\{1\}}(G) \cap Q_{\{2\}}(G) \cap Q_{\{3\}}(G) =$$
$$(x_1x_2 + y_1y_2, x_1x_3 + y_1y_3, x_1x_4 + y_1y_4, x_2x_3 + y_2y_3, x_2x_4 + y_2y_4, x_3x_4 + y_3y_4,$$
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Corollary *Let K be a field such that* $\mathrm{char}(K) \not\equiv 1, 2 \mod 4$ *or* $\mathrm{char}(K) = 0$. *Then the ideal L_G is prime if and only if G is a disjoint union of edges and isolated vertices.*

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Corollary *Let K be a field such that* $\operatorname{char}(K) \not\equiv 1, 2 \bmod 4$ *or* $\operatorname{char}(K) = 0$. *Then the ideal $L_{\overline{G}}$ is prime if and only if G is $(n-2)$-connected. In this case, $L_{\overline{G}}$ is a complete intersection.*

Question *Let $d \geq 1$, K be an arbitrary field and G an $(n-d)$-connected graph on $[n]$. Is $L_{\overline{G}}$ prime and a complete intersection ?*

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- Experimental data suggests that this could be true in general.
- **FALSE:** for primality. Counterexample for $d = 4$.
- It is true that $L_{\overline{G}}$ is prime and a complete intersection for $d \to \infty$. (Ongoing work with Aldo Conca)

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Corollary (ongoing work with A. Conca) *For any graph G the ideal of 3x3-minors of X_G is radical. It is prime if and only if G is $(n-2)$-connected.*

The END!