Splits and k-Splits of the Hypersimplex

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January 20, 2017
Osaka, Japan

Algebraic and Geometric Combinatorics on Convex Polytopes

Matroids

A (d, n)-*matroid* assigns to each subset of $S \subset [n]$ a *rank* $\mathrm{rk}(S) \in \mathbb{Z}$, s.t.

$$\mathrm{rk}(\emptyset) = 0 \qquad \text{and} \qquad \mathrm{rk}([n]) = d$$
$$\mathrm{rk}(S) \;\leq\; \mathrm{rk}(S \cup e) \;\leq\; \mathrm{rk}(S) + 1 \quad \text{for } e \in [n]$$
$$\mathrm{rk}(S) + \mathrm{rk}(S \cup f \cup g) \;\leq\; \mathrm{rk}(S \cup f) + \mathrm{rk}(S \cup g) \quad \text{for } f, g \in [n]$$

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A d-set $S \subset [n]$ is a *basis* if $\mathrm{rk}(S) = d$.

Example

- The rank function of the *uniform matroid* $U_{d,n}$ is $\mathrm{rk}(S) = \min\{\# S, d\}$. All d-sets are bases in this case.
- For a list of vectors $v_1, \ldots, v_n$ the following defines a rank function

$$\mathrm{rk}(S) = \dim \operatorname{span}\{v_i : i \in S\} \ .$$

The corresponding matroid is called *realizable*.

Matroid Polytopes

The convex hull of the characteristic vectors of the bases of a matroid M is the *matroid polytope*. For the uniform matroid this is the *hypersimplex*

$$\Delta(d,n) \;=\; \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = d \text{ and } x_i \geq 0 \right\} \;.$$

An outer describtion for the matroid polytope of M is

$$P_M \;=\; \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i \in F} x_i \leq \operatorname{rk}(F) \text{ for all subsets } F \text{ of } [n] \right\} \;.$$

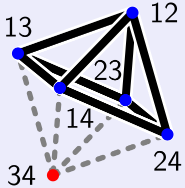
A set F is a *flacet* if the corresponding inequality is facet defining.

Example

The matroid of the columns of

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

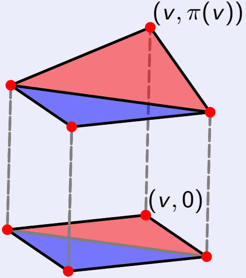
has 5 bases and $\operatorname{rk}(34) = 1$



The Dressian

We lift the vertices of $\Delta(d,n)$ with a height function $\pi \in \mathbb{R}^{\binom{n}{d}}$.
Liftings that decompose $\Delta(d,n)$ into matroid polytopes are called *tropical Plücker vectors* or *valuations* on the uniform matroid. Tropical Plücker vectors correspond to tropical linear spaces.

Example



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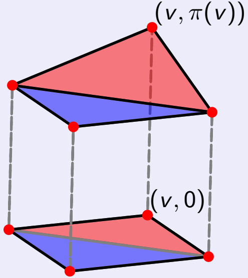
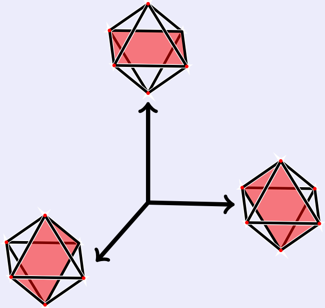
The *Dressian* $\mathrm{Dr}(d,n)$ is the subfan of the secondary fan of tropical Plücker vectors. This is the moduli space of all regular tropical linear spaces.

Example





The Dressian

The Dressian has been studied by various people.

(Fink, Herrman, Jensen, Joswig, Kapranov, Rincon, Speyer, Sturmfels ...)

- $\mathrm{Dr}(2, n)$ is the space of phylogenetic trees.
- For $d > 2$ only $\mathrm{Dr}(3, 6)$, $\mathrm{Dr}(3, 7)$ and $\mathrm{Dr}(3, 8)$ are completely known.
- All the rays of the above examples correspond to tropicalizations of linear spaces.
- A tropical point configuration gives rise to a matroid subdivision – this is obtained from a tropicalization of a linear space. (Stiefel map)
- The dimension of $Dr(3, n)$ is in $\Theta(n^2)$, that implies that 'most' points in the Dressian are not tropicalizations of linear spaces.

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Questions

- *What is the dimension of* $\mathrm{Dr}(d, n)$*?*
- *Are all of the rays of* $\mathrm{Dr}(d, n)$ *tropicalizations of linear spaces?*

Answers via corank vectors of split matroids

Theorem (Joswig, S. 2016)

The 'corank vector' of a 'split matroid' M is contained in the interior of a simplicial cone in $Dr(d, n)$. Moreover,

$$\frac{1}{n}\binom{n}{d} - 1 \;\leq\; \dim \mathrm{Dr}(d, n) \;\leq\; \binom{n-2}{d-1} - 1 \;.$$

For fixed d the dimension of the Dressian is of order $\Theta(n^{d-1})$.

Theorem (Joswig, S. 2016)

Let M be a connected split (d, n)-matroid. The 'corank vector' of the 'series-free lift' of M is a ray in the Dressian $\mathrm{Dr}(d+1, n+2)$. This ray corresponds to the tropicalization of a linear space if and only if M is realizable.

Canonical Subdivisions and Corank Vectors

Given a matroid M, the *corank-function* $\rho(M)$ is the height function that lifts the vertex of $\Delta(d, n)$ corresponding to S to height $d - \mathrm{rk}(S)$.

Speyer 2005: The vector $\rho(M)$ is a tropical Plücker vector.

The polytope P_M occurs as a cell in the induced subdivision.

Feichtner, Sturmfels 2005: This cell is maximal if and only if M is connected.

Lemma (Speyer 2005; Joswig, S. 2016)

The vector $\rho(M)$ corresponds to a tropicalization of a linear space if and only if M is realizable.

Splits

A split of $\Delta(d,n)$ is a subdivision with exactly two maximal faces. Two split-hyperplanes are compatible if they do not intersect in the interior of $\Delta(d,n)$.

Lemma (Joswig, S. 2016)

- *The facets of P_M are either supported by hypersimplex splits or are hypersimplex facets.*
- *Two flacets F and G are compatible if and only if*

$$\#(F \cap G) + d \leq \mathrm{rk}(F) + \mathrm{rk}(G) \ .$$

We call a matroid a *split matroid* if its flacets form a compatible system of hypersimplex splits.

Split and Paving Matroids

A matroid is *paving* if its rank function satisfies

$$\mathrm{rk}(S) = \# S \quad \text{for all sets } S \text{ with } \# S \leq d-1 \, .$$

It is conjectured that almost all matroids are paving.

Theorem (Joswig, S. 2016)

A connected (d, n)-matroid M is paving if and only if it is a split matroid such that each split flacet has rank $d-1$.

Remark: Split matroids are closed under dualization.
There is a excluded-minor characterization for the class of split matroids.

The percentage of paving matroids among the isomorphism classes of all matroids of rank d on n elements

$d \backslash n$	4	5	6	7	8	9	10	11	12
2	57	46	43	38	36	33	32	30	29
3	50	31	24	21	21	30	52	78	91
4	100	40	22	17	34	77	–	–	–
5		100	33	14	12	63	–	–	–
6			100	29	10	14	–	–	–
7				100	25	7	17	–	–
8					100	22	5	19	–
9						100	20	4	16
10							100	18	3
11								100	17

This computation has been done with `polymake`.

It is based on data from Matsumoto, Moriyama, Imai and Bremner.

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$d \backslash n$	4	5	6	7	8	9	10	11	12
2	100	100	100	100	100	100	100	100	100
3	100	100	89	75	60	52	61	80	91
4	100	100	100	75	60	82	–	–	–
5		100	100	100	60	82	–	–	–
6			100	100	100	52	–	–	–
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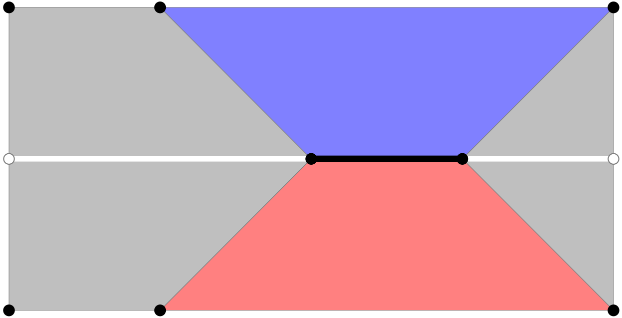
upper bound: Speyer's f-vector bound on the number of vertices of a tropical linear space.

lower bound: Knuth's construction for a stable set in the vertex-edge-graph of $\Delta(d,n)$ – a Johnson graph – gives a compatible set of vertex splits.

Rays in the Secondary Fan



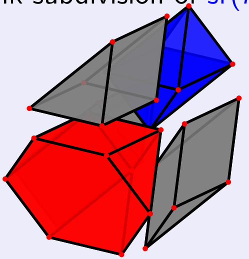
Idea: Embed the point configuration P and its subdivision into $P \times [-1, 1]$.
In our situation take $\Delta(d, n) \times \Delta(1, 2) \subset \Delta(d + 1, n + 2)$.
The construction uses the choice of a maximal cell. The hourglass shape reduces the dimension.

Rays of the Dressian

Let M be a connected split matroid and $\mathrm{sf}(M)$ the $(d+2, n+2)$-matroid, that is the series-extension at f of the free extension $M +_{[n]} f$.

Example

$(2, 6)$-matroid M with 3 split flacets. The corank subdivision of $\mathrm{sf}(M)$.



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- $P_M \times P_{U_{1,2}}$ is a facet of $P_{\mathrm{sf}(M)}$.
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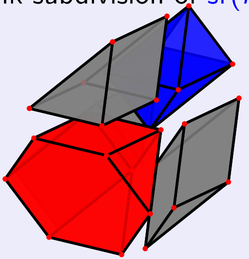
- The dual graph is 2-connected.
- All cycles in the dual graph have length 3.



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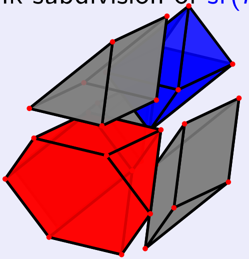

Nested Matroids

A (d,n)-matroid N is called nested if its split flacets form a chain $F_1 \subsetneq \cdots \subsetneq F_{k-1}$. The matroid polytope of N is given by

$$P_N \;=\; \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i \in F_\ell} x_i \leq \operatorname{rk}(F_\ell), \text{ for all } 1 \leq \ell \leq k-1 \right\} \; .$$

All flacets of N are incompatible if $k > 2$.

Example

The series-parallel lift $\mathrm{sf}^k(U_{1,2})$ is a nested matroid.
This matroid has $k-1$ split flacets.
The bounded part of the corresponding tropical linear space is a k-simplex.

k-Splits

A k-split is a subdivision of a point configuration into k maximal cells which intersect in an inner face of codim. $k-1$. The dual of a k-split is a k-simplex.

Theorem (Herrmann 2009)

A k-split is a coarsest regular subdivision and induced by a polyhedral fan with k rays and k maximal cells.

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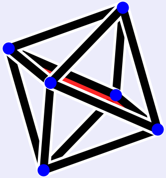

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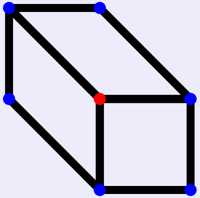
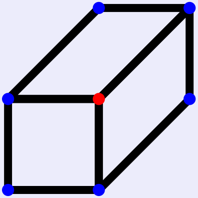
Splits are the same as 2-splits. Two examples of 3-splits:





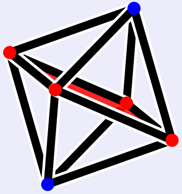
k-Splits of $\Delta(d, n)$ and $\Delta_{d-1} \times \Delta_{n-d-1}$

Let e_I be a vertex of $\Delta(d, n)$, that is in the intersection of all maximal cells of a k-split. Each such maximal cell intersects the vertex figure

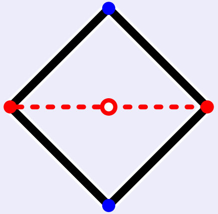
$$\mathcal{F}(e_I) = \left\{ x \in \Delta(d, n) \;\middle|\; \sum_{i \in I} x_i = d - 1 \right\}$$

full dimensional. The vertex figure is isomorphic to the product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$. Interior points of the induced subdivision are redundant.

Example



$\Delta(2, 4)$



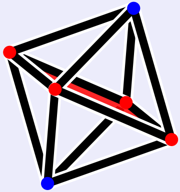
$\Delta_1 \times \Delta_1$

Consequences: Classification of all k-splits in $\Delta(d, n)$

Theorem (S. 2017$^+$)

Each k-split of the hypersimplex $\Delta(d, n)$ is a matroid subdivision.
The k maximal cells are connected nested matroids with $k-1$ split flacets or equivalently $k+1$ cyclic flats.
Moreover, this subdivision is realizable by a tropical point configuration, i.e., an image under the Stiefel map.

Example



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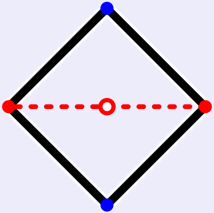

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Next Step

Express other subdivisions as combinations of k-splits.