Ehrhart Positivity

Federico Castillo

University of California, Davis

Joint work with Fu Liu

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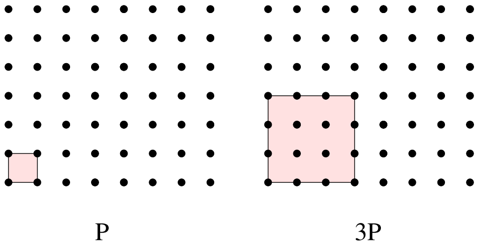
Definition

For any polytope $P \subset \mathbb{R}^d$ and positive integer $m \in \mathbb{N}$, the mth dilation of P is $mP = \{mx \ : x \in P\}$. We define

$$i(P, m) = |mP \cap \mathbb{Z}^d|$$

to be the number of lattice points in the mP.

Example



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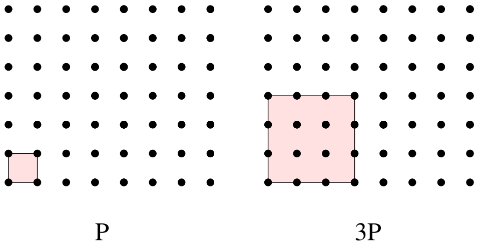

In this example we can see that $i(P, m) = (m+1)^2$

Theorem of Ehrhart (on integral polytopes)

Figure: Eugene Ehrhart.

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Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P.

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$$i([0,1]^3, m) = (m+1)^3,$$
$$i([0,1]^3, m) = m^3 + 3m^2 + 3m + 1,$$
$$i([0,1]^3, m) = 1\binom{m+3}{3} + 4\binom{m+2}{3} + 1\binom{m+1}{3} + 0\binom{m}{3}$$

The h^* or δ vector.

An alternative basis

We can write:

$$i(P,m) = h_0^*(P)\binom{m+d}{d} + h_1^*(P)\binom{m+d-1}{d} + \cdots + h_d^*(P)\binom{m}{d}.$$

The vector $(h_0^*, h_1^*, \cdots, h_d^*)$ has many good properties.

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Theorem(Stanley)

For any lattice polytope P, $h_i^*(P)$ is nonnegative integer.

Additionally it has an algebraic meaning.

Back to coefficients of Ehrhart polynomials

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No simple forms known for other coefficients for general polytopes.

Warning

It is NOT even true that all the coefficients are positive.

For example, for the polytope P with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(1, 1, 13)$, its Ehrhart polynomial is

$$i(P, n) = \frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1.$$

By the way...

Theorem (A.Tsuchiya)

Other than the leading, second, and constant being positive, **any** sign pattern can occur in the other coefficients.

General philosophy.

They are related to volumes.

Ehrhart Positivity

Main Definition.

We say an integral polytope is *Ehrhart positive* (or just positive for this talk) if it has positive coefficients in its Ehrhart polynomial.

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In the literature, different techniques have been used to proved positivity.

Example I

Polytope: Standard simplex.

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Reason: Explicit verification.

Standard simplex.

In the case of

$$\Delta_d = \{\mathbf{x} \in \mathbb{R}^{d+1} : x_1 + x_2 + \cdots + x_{d+1} = 1, x_i \geq 0\},$$

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$$\binom{m+d}{d} = \frac{(m+d)(m+d-1)\cdots(m+1)}{d!}$$

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$$\binom{m+d}{d} = \frac{(m+d)(m+d-1)\cdots(m+1)}{d!}$$

which expands positively in powers of m.

Hypersimplices.

In the case of

$$\Delta_{d+1,k} = \operatorname{conv}\{\mathbf{x} \in \{0,1\}^{d+1} : x_1 + x_2 + \cdots + x_{d+1} = k\},$$

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$$\sum_{i=0}^{d+1} \binom{d+1}{i} \binom{d+1+mk-(m+1)i-1}{d} (-1)^i$$

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Not clear if the coefficients are positive.

Example II

Polytope: Crosspolytope

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Reason: Roots have negative real part.

Crosspolytope.

In the case of the crosspolytope:

$$\Diamond_d = \mathrm{conv}\{\pm e_i : 1 \leq i \leq d\},$$

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$$(n + 1/2) \quad \text{or} \quad (n + 1/2 + ia)(n + 1/2 - ia) = n^2 + n + 1/4 + a^2,$$

where a is real, so positivity follows.

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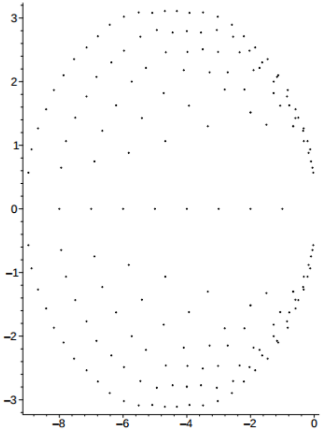
What are the roots about?

This opens more questions.

Birkhoff Poytope

The following is the graph (Beck-DeLoera-Pfeifle-Stanley) of zeros for the Birkhoff polytope of 8×8 doubly stochastic matrices.



Example III

Polytope: Zonotopes.

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Reason: Formula for them.

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One of the few examples in which the formula is explicit on the coefficients.

Zonotopes.

Definition

The Minkowski sum of vectors

$$\mathcal{Z}(v_1, \cdots, v_k) = v_1 + v_2 + \cdots + v_k.$$

The Ehrhart polynomial

$$i(\mathcal{Z}(v_1, \cdots, v_k), m) = a_d m^d + a_{d-1} m^{d-1} + \cdots a_0 m^0,$$

has a coefficient by coefficient interpretation.

Zonotopes.

Theorem(Stanley)

In the above expression, a_i is equal to (absolute value of) the greatest common divisor (g.c.d.) of all $i \times i$ minors of the matrix

$$M = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & \cdots & | \end{bmatrix}$$

Zonotopes.

This includes the unit cube $[0, 1]^d$ which has Ehrhart polynomial

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$$i(\square_d, m) = (m+1)^d.$$

And also the regular permutohedron

$$\begin{aligned}\Pi_n &= \sum_{1 \le i < j \le n+1} [e_i, e_j], \\ &= \mathrm{conv}\{\Big(\sigma(1), \sigma(2), \cdots, \sigma(n+1)\Big) \in \mathbb{R}^{n+1} : \sigma \in S_{n+1}\}.\end{aligned}$$

Permutohedron.

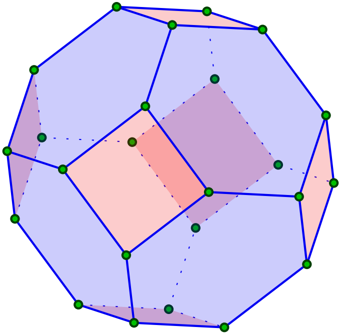

Figure: A permutohedron in dimension 3.

The Ehrhart polynomial is $1 + 6m + 15m^2 + 16m^3$.

Example IV

Polytope: Cyclic polytopes.

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Reason: Higher integrality conditions.

Cyclic polytopes.

Consider the moment map $m : \mathbb{R} \to \mathbb{R}^d$ that sends

$$x \mapsto (x, x^2, \cdots, x^d).$$

The convex hull of any(!) n points on that curve is what is called a cyclic polytope $C(n, d)$.

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Ehrhart Polynomial.

Fu Liu proved that under certain integrality conditions, the coefficient of t^k in the Ehrhart polynomal of P is given by the volume of the projection that forgets the last k coordinates.

Not a combinatorial property

Theorem (Liu)

For any polytope P there is a polytope P' with the same face lattice and Ehrhart positivity.

Plus many unknowns.

Other polytopes have been observed to be positive.

- CRY (Chan-Robbins-Yuen).
- Tesler matrices (Mezaros-Morales-Rhoades).
- Birkhoff polytopes (Beck-DeLoera-Pfeifle-Stanley).
- Matroid polytopes (De Loera - Haws- Koeppe).

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Also:

Littlewood Richardson

Ronald King conjecture that the *stretch* littlewood richardson coefficients $c_{t\lambda,t\mu}^{t\nu}$ are polynomials in $\mathbb{N}[t]$. This polynomials are known to be Ehrhart polynomials.

General approach?

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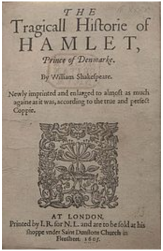
“Though it be madness, yet there’s method in’t...” Hamlet, Act II.



THE
Tragicall Hiſtorie of
HAMLET,
Prince of Denmarke.

By William Shakeſpeare.

Newly imprinted and enlarged to almoſt as much againe as it was, according to the true and perfect Coppie.

AT LONDON,
Printed by I. R. for N. L. and are to be ſold at his ſhoppe vnder Saint Dunſtons Church in Fleetſtreet. 1605.

Method in the madness.

Coming from the theory of toric varieties, we have

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Definition

A *McMullen* formula is a function α such that

$$|P \cap \mathbb{Z}^d| = \sum_{F \subseteq P} \alpha(F, P)\mathrm{nvol}(F).$$

where the sum is over all faces and α depends locally on F and P. More precisely, it is defined on the normal cone of F in P.

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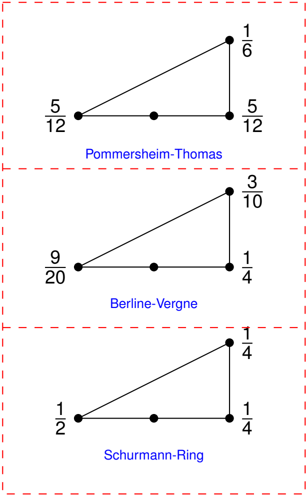
McMullen proved the existence of such α in a nonconstructive and nonunique way.

Constructions

There are at least three different constructions

1. Pommersheim-Thomas. Need to choose a flag of subspaces.
2. Berline-Vergne. No choices, invariant under $O_n(\mathbb{Z})$. This is what we use.
3. Schurmann-Ring. Need to choose a fundamental cell.

Example



McMullen Formula:

$$|P \cap \mathbb{Z}| = (\text{Area of P}) + \frac{1}{2}(\text{Perimeter of P}) + 1.$$

The way one gets the +1 is different.

Refinement of positivity.

This gives expressions for the coefficients.

$$
\begin{aligned}
|nP \cap \mathbb{Z}^d| &= \sum_{F \subset nP} \alpha(F, nP)\mathrm{nvol}(F) \\
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Coefficient

The coefficient of n^k is $\sum_{F:\dim(F)=k} \alpha(P, F) vol(F)$.

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As long as all α are positive, then the coefficients will be positive.

Main properties.

The important facts about the Berline-Vergne construction are

- It exists.
- Symmetric under rearranging coordinates.
- It is a valuation.

We exploit these.

A refined conjecture.

We pose the following.

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The regular permutohedron is (Berline-Vergne) α positive.

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We care because this imply Ehrhart positivity for a family of polytopes.

Proposition.

The above conjecture implies that Generalized Permutohedra are positive.

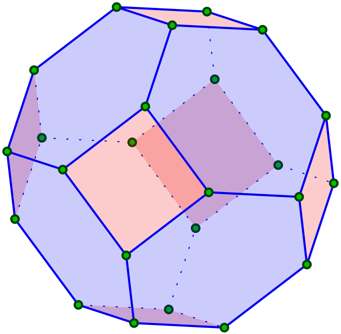
This would expand on previous results from Postnikov, and a conjecture of De Loera-Haws-Koeppe stating that matroid polytopes are positive.

Partial results.

We've checked the conjecture in the cases:

1. The linear term (corresponding to edges) in dimensions up to 100.
2. The third and fourth coefficients.
3. Up to dimension 6.

Regular permutohedra revisited.



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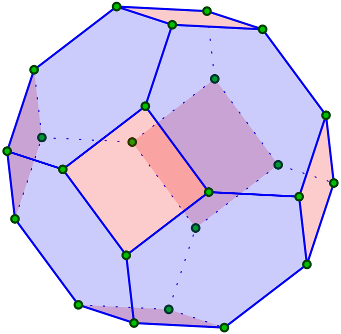
Regular permutohedra revisited.



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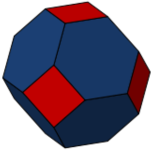
For example, $\alpha(v, \Pi_3) = \frac{1}{24}$ for any vertex. Since they are all symmetric and they add up to 1.

A deformation.



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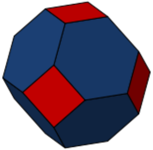

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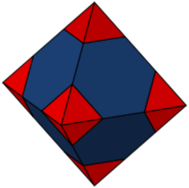

Computing with the properties.

Note that we have just two types of edges (with normalized volume 1). From the permutohedron we get

$$24\alpha_1 + 12\alpha_2 = 6.$$

Now looking at the octhaedron, the alpha values are the same, since the normal cones didn't change. In this case we get

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Remark.

We did not use the explicit construction at all, just existence and properties. This line of thought is the one we generalize.

A computation

- There is just one 3 dimensional face, with $\alpha = 1$ and volume $4^{4-2} = 16$ which contributes $\mathbf{16n^3}$
- There are six 2 dimensional faces with volume 1 and eight with volume 3. The α value is $1/2$ for all these faces so we get a contribution of $6\frac{1}{2}1 \cdot n^2 + 8\frac{1}{2}3 \cdot n^2 = \mathbf{15n^2}$
- Two types of edges, both with volume 1. There are 24 short edges with value $11/72$ and 12 long edges with value $14/72$, for a contribution of $24\frac{11}{72}n + 12\frac{14}{72}n = \mathbf{6n}$
- There are 24 vertices, all with value $1/24$ with a contribution of **1**

Putting all together we get the Ehrhart polynomial

$$1 + 6n + 15n^2 + 16n^3$$

Main result.

We have a combinatorial formula for the α values of faces of regular permutohedra. This formula involves *mixed Ehrhart coefficients of hypersimplices.* The takeaway from this is

Uniqueness theorem.

Any McMullen formula that is symmetric under the coordinates is uniquely determined on the faces of permutohedra.

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Uniqueness theorem.

Any McMullen formula that is symmetric under the coordinates is uniquely determined on the faces of permutohedra.

Which leads to the question.

Question.

Is Berline and Vergne the only construction that satisfies additivity and symmetry?

Warning

We want to remark that it is **not** true that zonotopes are BV α positive, even though they are Ehrhart positive.

A bit about the formula

Let $P_1, \cdots, P_m$ be a list of polytopes of dimension n, then

Mixed Valuations

The expression $\mathrm{Lat}(w_1 P_1 + \cdots + w_m P_m)$ is a polynomial on the w_i variables. The coefficients are called *mixed Ehrhart* coefficients.

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On the top degree we have the mixed volumes.

A bit about the formula

Let $P_1, \cdots, P_m$ be a list of polytopes of dimension n, then

Mixed Valuations

The expression $\mathrm{Lat}(w_1 P_1 + \cdots + w_m P_m)$ is a polynomial on the w_i variables. The coefficients are called *mixed Ehrhart* coefficients.

On the top degree we have the mixed volumes. Volumes are always positive and mixed volumes are too, although this is not clear from the above definition.

Permutohedra

We define a permutohedron for any vector $\mathbf{x}) = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}$. Let's assume $x_1 \leq \cdots \leq x_{n+1}$.

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So the number of integer points depends polynomially on the parameters w_i.These parameters are the lenghts of the edges in Perm($\mathbf{x}$).
For instance, the coefficient of $w_1 w_2$ is, by definition,

$$2!\mathrm{MLat}^2(\Delta_{1,n+1}, \Delta_{2,n+1})$$

Formula

Roughly

What we have looks like

$$\alpha(F, P) = A \times B.$$

Where A is some combinatorial expression, evidently positive.
And B is one (depending of F) mixed Ehrhart coefficient of *hypersimplices*.

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In particular, our conjecture is equivalent to the positivity of such coefficients. It is not even clear if hypersimplices themselves (without any mixing) are Ehrhart positive.

Example

An instance of the formula looks like:

A facet in Π_3

Formula would say it is equal to

$$\frac{2 \cdot 2}{24} 2! \mathrm{MLat}^2(\Delta_{1,4}, \Delta_{3,4}).$$

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This mixed valuations can be evaluated in the usual alternating form.
We can check if the above expression is right. Let's do it!

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How to find that coefficient? Inclusion-Exclusion!

Example

$$\begin{aligned} i(\Delta_{14}+\Delta_{34},t) &= \frac{10}{3}t^3+5t^2+\frac{11}{3}t+1, \\ i(\Delta_{14},t) &= \frac{1}{6}t^3+t^2+\frac{11}{6}t+1, \\ i(\Delta_{34},t) &= \frac{1}{6}t^3+t^2+\frac{11}{6}t+1. \end{aligned}$$

Therefore,

$$2!\mathrm{MLat}^2(\Delta_{1,4},\Delta_{3,4}) = 5-1-1=3$$

So we get

$$\frac{2\cdot 2}{24}\,2!\mathrm{MLat}^2(\Delta_{1,4},\Delta_{3,4}) = \frac{4}{24}\cdot 3 = \frac{1}{2}$$

Further direction.

Some observations lead to the very natural question:

Sum of positives.

If P and Q are positive, is it true that $P + Q$ is positive?


ありがとう