Transfer-Matrix Methods Meet Ehrhart Theory

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Outline

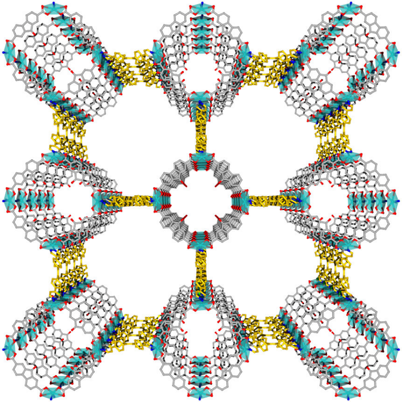

Figure: A Cartesian graph product in the wild (Credit: Rutgers University)

Outline

- A graph G is a pair (V, E), where V is the set of **nodes** and where E is the set of **edges**.

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- We denote the path graph on n vertices by P_n and the cycle graph on n vertices by C_n.

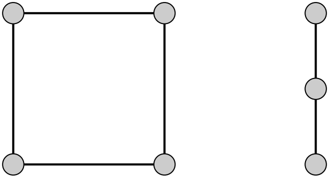

Figure: C_4 and P_3

- A graph G is a pair (V, E), where V is the set of **nodes** or vertices and where E is the set of **edges**.
- We denote the path graph on n vertices by P_n and the cycle graph on n vertices by C_n.
- A *proper k-coloring of G* is a map $c\colon V \longrightarrow \{1, 2, \ldots, k\}$ such that $c(v) \neq c(u)$ for all u, v with $\{u, v\} \in E$.

Figure: A proper 2-coloring and a non-proper 2-coloring

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- As it turns out, χ_G is a polynomial of degree $N = \#V(G)$ with leading coefficient 1.
- $\chi_G(-1)$ counts the number of **acyclic orientations** of G. This is a special case of a reciprocity theorem for χ_G.
- We are interested in the number of proper k-colorings of $G \times P_n$ or $G \times C_n$, where k and n are variables.

- For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ the Cartesian product $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$
- The vertices (u_1, v_1) and (u_2, v_2) are connected by an edge if $(u_1 = u_2$ and $\{v_1, v_2\} \in E_2)$ or if $(v_1 = v_2$ and $\{u_1, u_2\} \in E_1)$.

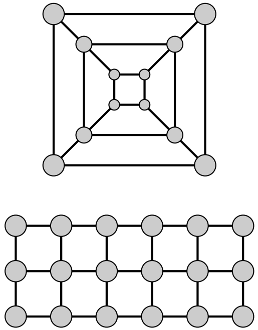

Figure: $C_4 \times P_3$ and $P_3 \times P_6$.

Question

Let G be any graph and let n and k be positive integers. How many proper k-colorings of $G \times P_n$ are there?

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What is the asymptotic behavior of the number of proper k-colorings of $G \times C_n$ as $k, n \to \infty$?

Outline

There are two special cases:

- k is fixed and n is not $\Longrightarrow$ Transfer-Matrix Method, see e. g. [8].
- n is fixed and k is not $\Longrightarrow$ Inside-Out Polytopes, see [1].

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Example

If $G = C_3$, then

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

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Transfer-Matrix Theorem

For a nonnegative integer n, $(A^n)_{i,j}$ counts the number of walks of length n from v_i to v_j.

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Example continued

If $G = C_3$, then

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

- We want to count the number of proper k-colorings of $G \times P_n$ for *fixed* k. To use the transfer-matrix method, we establish a connection between walks of length n and colorings of $G \times P_n$.

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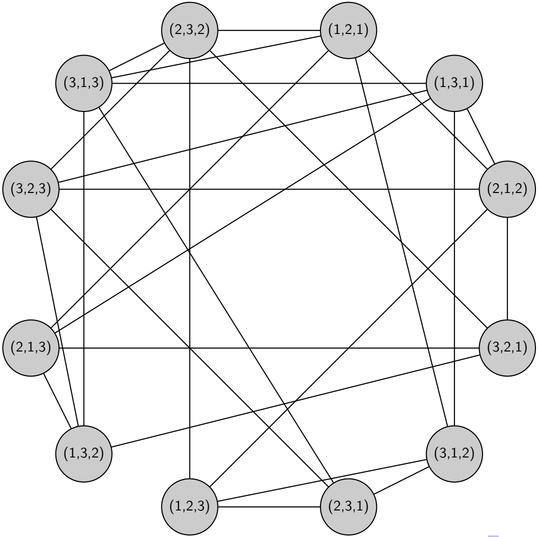
- We want to count the number of proper k-colorings of $G \times P_n$ for *fixed* k.
- We now associate a *new* graph M_G to G.
- The vertices of M_G are labeled by the proper k-colorings of G. Two vertices are connected by an edge if they form a proper coloring of $G \times P_2$.

Example

Let k be 3 and let $G = P_3$. Then M_G looks like:

The Transfer-Matrix Method



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- We can use the adjacency matrix A_{M_G} of M_G to count exactly these walks!
- Similarly, the number of closed walks ($=$ the number of colorings of $G \times C_{n+1}$) is counted by the trace of $A^n_{M_G}$
- Asymptotically, the trace is dominated by $\lambda_{\max}^{n-1}$, where $\lambda_{\max}$ is the biggest eigenvalue, which is real and positive by the Frobenius-Perron theorem.

	121	131	141	212	232	242	313	323	343	414	424	434	123	124	132	134	142	143	213	214	231	234	241	243	312	314	321	324	341	342	412	413	421	423	431	432
121	0	0	0	1	1	1	1	0	1	1	0	1	0	0	0	0	0	0	1	1	0	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1
131	0	0	0	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	1	0	1	1	1	0	1	0	0
141	0	0	0	1	1	0	1	1	0	1	1	1	0	0	0	0	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	1
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213	1	1	1	0	0	0	0	0	0	0	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	0	1	1
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321	0	0	0	1	1	1	0	0	0	1	0	1	0	0	1	1	1	1	1	1	0	1	0	1	0	0	0	0	0	0	1	1	0	0	0	1
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341	0	0	0	1	1	0	0	0	0	1	1	1	1	1	1	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0	1	1	0	1	0	1
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421	0	0	0	1	1	1	1	0	1	0	0	0	0	0	1	1	1	1	1	1	0	1	0	1	1	1	0	0	0	1	0	0	0	0	0	0
423	0	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	0	0	1	1	1	1	0	1	1	0	0	1	1	0	0	0	0	0	0
431	0	0	0	1	0	1	1	1	1	0	0	0	1	1	0	0	1	1	1	1	0	0	0	1	1	1	0	1	0	1	0	0	0	0	0	0
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Figure: The adjacency matrix for M_{P_3} for 4 colors

Conclusions

- The size of the matrix depends on $k \implies$ we can't directly use this technique for k not fixed.
- The size of the matrix is big, so even for fixed k this is computationally challenging.
- We will use symmetry to obtain a new matrix whose size does **not** depend on k and whose biggest eigenvalue is $\lambda_{\max}$.

Inside-Out Polytopes:

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- This has a beautiful interpretation in terms of Ehrhart theory and inside-out polytopes.

Inside-Out Polytopes:

- If n is fixed and k is a variable, then the number of proper colorings of $G \times P_n$ is counted by the chromatic polynomial.
- This has a beautiful interpretation in terms of Ehrhart theory and inside-out polytopes, see [1].

Example

Let $G = P_2$. Then the k-colorings are given by $\{(x, y) \colon x, y \in [k] \text{ and } x \neq y\}$.

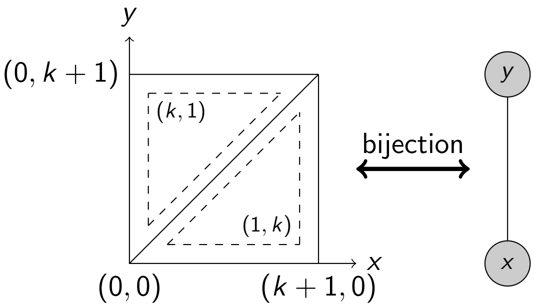

Figure: Integer points inside dashed triangles correspond to proper k-colorings of P_2.

- This works in general!
- If G is a graph with vertex set $V = \{x_1, \ldots, x_n\}$, then we get an **inside-out polytope** by taking

$$P_G := [0,1]^n \setminus \left(\bigcup H_{i,j} \right),$$

where we get a forbidden hyperplane $H_{i,j}$ if $\{x_i, x_j\} \in E$. We also denote this by $(P, \mathcal{H})$, where $\mathcal{H}$ is a collection of the hyperplanes.

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where we get a forbidden hyperplane $H_{i,j}$ if $\{x_i, x_j\} \in E$.

- Counting proper colorings $\leftrightarrow$ counting integer points.
$\Longrightarrow$ Enter Ehrhart theory.

- If P is a lattice polytope, one can define the counting function

$$E_P(t) := \#\left(tP \cap \mathbb{Z}^d\right),$$

which is called the *Ehrhart function* of P.

- If P is a lattice d-polytope, one can define the counting function
$$E_P(t) := \#\left(tP \cap \mathbb{Z}^d\right),$$
which is called the *Ehrhart polynomial* of P.
- Ehrhart famously proved that this is a polynomial of degree d with leading coefficient $\operatorname{vol}(P)$, see [3].

Theorem

Let P be a d-dimensional rational polytope. Then

$$E_P(-t) = (-1)^d E_{P^\circ}(t) \tag{1}$$

where $E_{P^\circ}(t)$ counts the number of integer points in the interior of tP.

Example

Let $P = [0,1]^2$. Then $E_P(t) = (t+1)^2$, and $E(2) = 9$ and $E(-2) = (-1)^2 \cdot 1$.

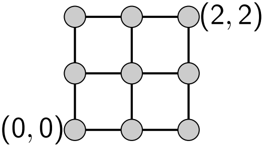
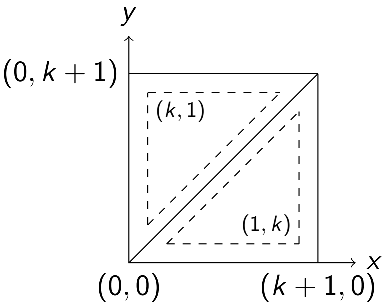


Figure: $2P = [0,2]^2$.

- Beck and Zaslavsky apply reciprocity to every piece and they thus get [1, Theorem 4.1]:

Theorem

Reciprocity works for inside-out polytopes, but you need to account for multiplicities of the integer points on the hyperplanes.



- The reason why the previous results are interesting for us is [1, Theorem 5.1]:

Theorem

For a graph G and the inside-out polytope P_G, we have

$$E_{P_G^\circ}(t) = \chi_G(t-1). \tag{2}$$

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- This theorem together with the reciprocity result gives us a geometric proof of Stanley's recipocity theorem!

Theorem

For a graph G and the inside-out polytope P_G, we have

$$E_{P_G^\circ}(t) = \chi_G(t-1). \tag{4}$$

- This theorem together with the reciprocity result gives us a geometric proof of Stanley's recipocity theorem!
- Now let's combine this geometric setup with the transfer-matrix method!

Outline

Example revisited

Let $G = P_3$. We want to count the number of proper k-colorings of $G \times P_n$, where n and k are variables.

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131	0	0	0	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	1	0	1	1	1	0	1	0	0
141	0	0	0	1	1	0	1	1	0	1	1	1	0	0	0	0	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	1
212	1	1	1	0	0	0	0	1	1	0	1	1	1	1	0	1	0	1	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	0
232	1	0	1	0	0	0	1	1	1	1	1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0	1	1	1	0	0
242	1	1	0	0	0	0	1	1	0	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	0
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343	1	1	0	1	1	0	0	0	0	1	1	1	0	1	1	1	0	0	0	1	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	1
414	1	1	1	0	1	1	0	1	1	0	0	0	1	0	1	0	1	1	0	0	1	0	1	1	0	0	1	0	1	1	0	0	0	0	0	0
424	0	1	1	1	1	1	1	0	1	0	0	0	0	0	1	0	1	1	1	0	1	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0
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124	0	0	0	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1	1	0	0	0	1	1	1	1	0	0	1	1
132	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	1	1	0	0	1	1	0	1	1	1	1	0	0	1	1	1	0	0
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142	0	0	0	0	0	0	1	1	0	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	1	0
143	0	0	0	1	1	0	0	0	0	1	1	1	0	0	0	0	0	0	0	1	1	1	0	0	1	1	1	1	0	0	1	0	1	0	1	1
213	1	1	1	0	0	0	0	0	0	0	1	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	0	1	1
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231	0	0	0	0	0	0	1	1	1	1	1	0	1	1	0	0	1	1	0	0	0	0	0	0	1	1	0	1	0	1	1	1	0	1	0	0
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431	0	0	0	1	0	1	1	1	1	0	0	0	1	1	0	0	1	1	1	1	0	0	0	1	1	1	0	1	0	1	0	0	0	0	0	0
432	1	0	1	0	0	0	1	1	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	1	1	1	1	0	0	0	0	0	0	0

Figure: The adjacency matrix for M_{P_3} for 4 colors

Example revisited

Let $G = P_3$. We want to count the number of proper k-colorings of $G \times P_n$, where n and k are variables.

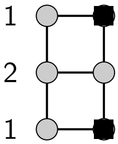
There are essentially two cases:

- v_1 and v_3 have the same color (orbit 1)
- v_1 and v_3 have different colors (orbit 2)

So let's define a matrix that encodes exactly this information. This matrix will be a 2×2 matrix.

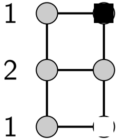
Let $A \in R^{2\times 2}$, where

- $a_{1,1}$ counts the number of proper k-colorings of $G \times P_2$, where the first copy of G is colored by a representative of orbit 1 and the second has to be in orbit 1.



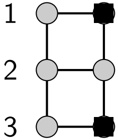
Let $A \in R^{2\times 2}$, where

- $a_{1,2}$ counts the number of proper k-colorings of $G \times P_2$, where the first copy of G is colored by a representative of orbit 1 and the second has to be in orbit 2.



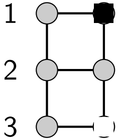
Let $A \in R^{2\times 2}$, where

- $a_{2,1}$ counts the number of proper k-colorings of $G \times P_2$, where the first copy of G is colored by a representative of orbit 2 and the second has to be in orbit 1.



Let $A \in R^{2\times 2}$, where

- $a_{2,2}$ counts the number of proper k-colorings of $G \times P_2$, where the first copy of G is colored by a representative of orbit 2 and the second has to be in orbit 2.



- This gives us that

$$A = \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}$$

  for $k \geq 3$.

- For $k = 4$, this simplifies to

$$A = \begin{pmatrix} 7 & 10 \\ 5 & 11 \end{pmatrix}.$$

For a general graph, there are two types of symmetries:

1. relabeling of the colors
2. graph automorphisms of G

Example

Let $G = P_4$. Using a relabeling of colors we get 5 cases.

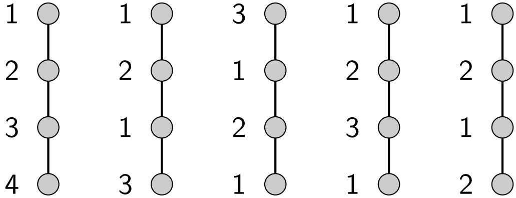
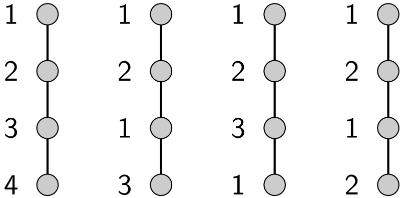


Figure: Representatives of the 5 orbits.

Example

Let $G = P_4$. Then only using a relabeling of colors gives us 5 cases. Also using graph symmetry gives us 4 orbits instead.



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Theorem

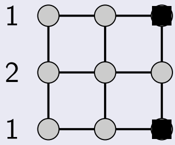
The (i, j)-entry of A^n counts the number of colorings of $G \times P_{n+1}$, where the first G is fixed by a representative of orbit o_i and the last G is colored by an element of orbit o_j.

Example continued

Let $G = P_3$. Recall that

$$A = \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}.$$

Then $(A^{2})_{1,1} = k^5 - 9k^4 + 36k^3 - 77k^2 + 87k - 41$ counts the number of colorings of



Example continued

Let $G = P_3$. Recall that

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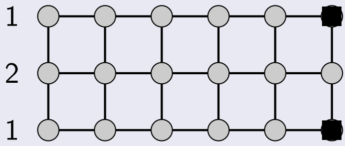


Corollary

Let $G \times P_{n+1}$ and A be as above. Then

$$\chi_{G\times P_{n+1}}(k) = (w_1(k), \ldots, w_p(k))A^n\mathbf{1}, \tag{5}$$

where we set $w_i(k)$ is the size of orbit i and $\mathbf{1} := (1, \ldots, 1)^t$.

This means that A contains all the information needed to get the chromatic polynomial for every n.

Example continued

The chromatic polynomial of $P_3 \times P_3$ is

$$\chi(k) = (w_1, w_2) \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}^{3-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $w_1 = k(k-1)$ and $w_2 = k(k-1)(k-2)$.

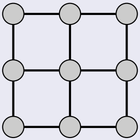


Figure: $P_3 \times P_3$

Example continued

The chromatic polynomial of $P_3 \times P_4$ is

$$\chi(k) = (w_1, w_2) \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}^{4-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $w_1 = k(k-1)$ and $w_2 = k(k-1)(k-2)$.

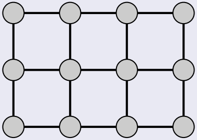


Figure: $P_3 \times P_4$

Example continued

The chromatic polynomial of $P_3 \times P_5$ is

$$\chi(k) = (w_1, w_2) \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}^{5-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $w_1 = k(k-1)$ and $w_2 = k(k-1)(k-2)$.

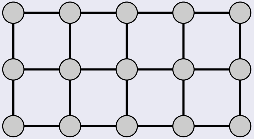


Figure: $P_3 \times P_5$

Example continued

The chromatic polynomial of $P_3 \times P_6$ is

$$\chi(k) = (w_1, w_2) \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}^{6-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $w_1 = k(k-1)$ and $w_2 = k(k-1)(k-2)$.

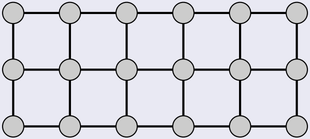


Figure: $P_3 \times P_6$

Proposition

Let $\lambda_{\max}$ be the biggest eigenvalue of A (and thus of the adjacency matrix A_{M_G}). Then

$$\delta(A) \leq \lambda_{\max} \leq \Delta(A), \tag{6}$$

where $\delta(A)$ and $\Delta(A)$ are polynomials of degree N and their two highest coefficients agree. We also have a combinatorial interpretation for these coefficients.

Example continued

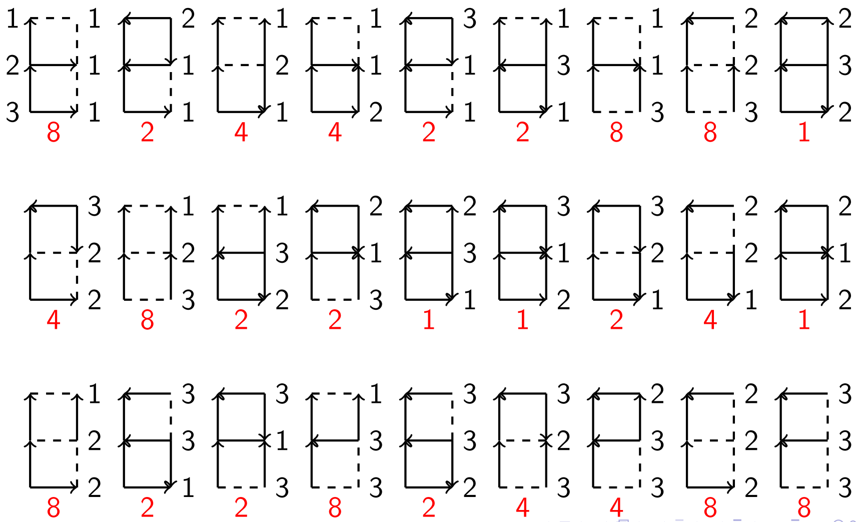
Let $G = P_3$. The biggest eigenvalue $\lambda_{\max}$ of A satisfies the inequalities

$$k^3 - 5k^2 + 10k - 8 \leq \lambda_{\max} \leq k^3 - 5k^2 + 10k - 7.$$

The asymptotic behavior of the proper k-colorings of $P_3 \times C_{n+1}$ is dominated by $\lambda_{\max}^n$.

Theorem

The row sums of A^n satisfy a restricted reciprocity theorem for $k \geq N$.


1 2 3
1 1 1 8
2 1 1 2
1 2 1 4
1 1 2 4
3 1 1 2
1 3 1 2
1 1 3 8
2 2 3 8
2 3 2 1
3 2 2 4
1 2 3 8
1 3 2 2
2 1 3 2
2 3 1 1
3 1 2 1
3 2 1 2
2 2 1 4
2 1 2 1
1 2 2 8
3 3 1 2
3 1 3 2
1 3 3 8
3 3 2 2
3 2 3 4
2 3 3 4
2 2 2 8
3 3 3 8

Outline

- If we do not use graph automorphisms, we can count the number of orbits with a deletion-contraction process. This led to several (new?) recursions for Bell numbers.

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- The same philosophy can be applied to many more objects, e.g., Discrete Markov Processes, and what we call **Stacked Posets**.
- The underlying geometry of inside-out polytopes is behind all of the theorems and proofs.

Thanks for your attention!

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