On the maximal dual volume of a lattice polytope with one interior point

and many other amazing facts about Ehrhart Theory and the volume of lattice polytopes

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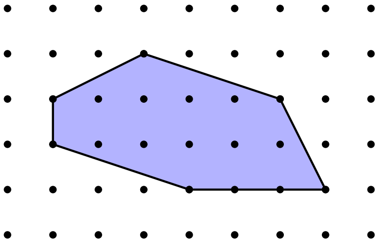
Osaka

$P \subset \mathbb{R}^d$ full-dimensional lattice **polytope** with respect to the lattice $\mathbb{Z}^d$.



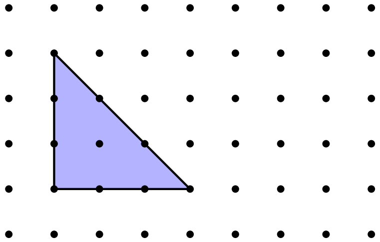
Combinatorial objects with **many** beautiful connections with other subjects:

- algebraic geometry (toric geometry)
- optimization (integer programming)
- commutative algebra (semigroup algebras)
- theoretical physics (mirror symmetry)
- ...

It's important to study relations among their invariants!

Some invariants we study:

- Lattice points (how many and where)
- Volume
- Faces (vertices,edges,...)
- Triangulations
- Degree
- Ehrhart polynomial
- h^*-polynomial $\leftarrow$ my favorite!
- Width
- ...



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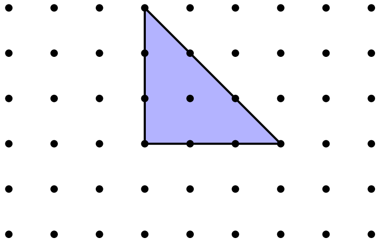

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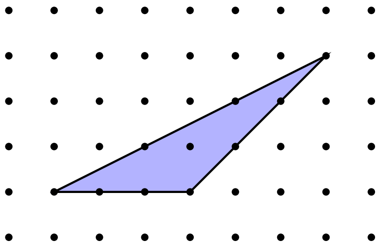

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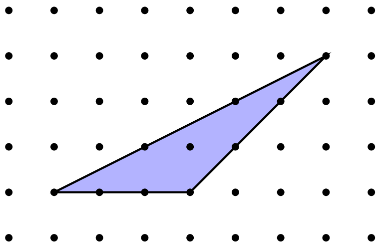

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So we consider lattice polytopes up to affine automorphisms of the lattice.

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$$\int_P 1 d\mu$$

- μ standard d-dimensional Lebesgue measure $\to \mathrm{vol}(P)$
- μ "counting measure" $X \mapsto \left|X \cap \mathbb{Z}^d\right| \to \left|P \cap \mathbb{Z}^d\right|$

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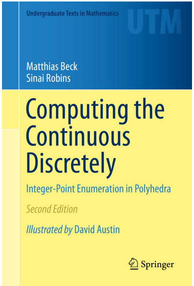
$$\left|P \cap \mathbb{Z}^d\right| \overset{?}{\longleftrightarrow} \mathrm{vol}(P)$$



$\boxed{\text{dim } 1}$ EASY!

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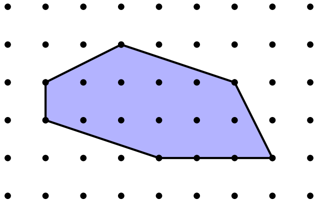
$\boxed{\text{dim } 2}$ EASY thanks to Pick's Formula!



$$\mathrm{vol}(P) = |P^\circ \cap \mathbb{Z}^d| + \frac{1}{2}|\partial P \cap \mathbb{Z}^d| - 1$$

$$12 \quad = \quad 9 \quad + \quad 4 \quad - 1$$

Proved by Pick in 1899. It works also for non-convex polygons!

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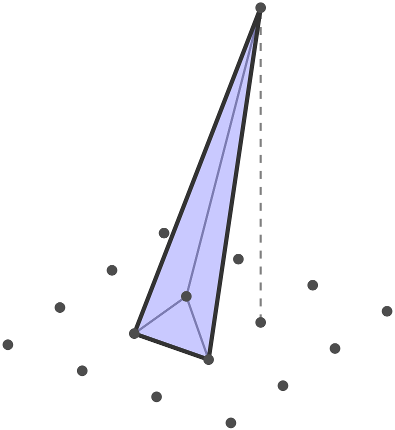

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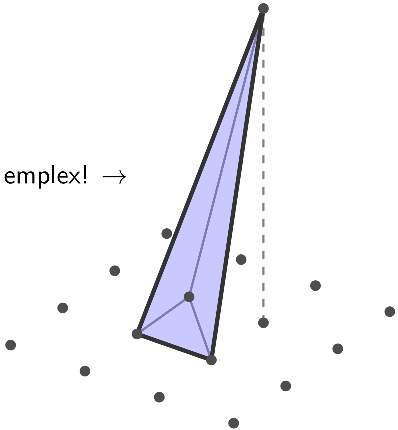

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If P is a d-dimensional lattice polytope,we define the function

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THEOREM (EHRHART 1962)

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$$\mathrm{ehr}_P(k) = c_d k^d + c_{d-1} k^{d-1} + \cdots + c_1 k + c_0$$

- $c_0 = 1$
- $c_d = \mathrm{vol}(P)$
- $c_{d-1} = \frac{\mathrm{vol}(\partial P)}{2}$
- the other coefficients do not have a totally understood combinatorial interpretation (think of Federico's talk)

If we want to know $\mathrm{vol}(P)$, we need to count the lattice points in P,$2P$,...,dP.

Question

Can one characterize all polynomials which are Ehrhart polynomial of some lattice polytope?

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- trivial in dimension 1
- solved by Scott in dimension 2 (1976)
- totally open in dimension ≥ 3

One may rephrase the previous question by considering the corresponding generating series of ehr_P

$$\sum_{k \geq 0} \mathrm{ehr}_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}}$$

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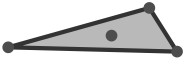

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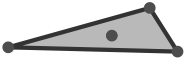

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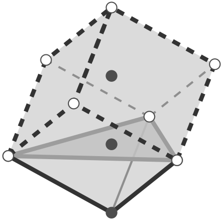

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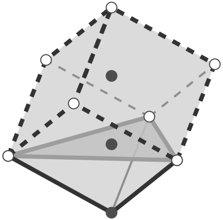

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This can be extended from simplices to polytopes via half-open triangulations.

All the coefficients have a combinatorial interpretation!

- $h_i^* \in \mathbb{Z}_{\geq 0}$ for each i
- $h_0^* = 1$
- $h_1^* = |P \cap \mathbb{Z}^d| - d - 1$
- $h_d^* = |\mathrm{int}(P) \cap \mathbb{Z}^d|$

Moreover

- $1 + h_1^* + \ldots + h_d^* = d!\mathrm{vol}(P) =: \mathrm{Vol}(P)$ normalized volume
- monotonicity $P \subseteq Q$, $h_{P,i}^* \leq h_{Q,i}^*$ for each i

Some necessary conditions are already known.

- $h_1^* \geq h_d^*$ (Ehrhart)
- $h_d^* + h_{d-1}^* + \cdots + h_{d-i}^* \leq h_0^* + h_1^* + \cdots + h_{i+1}^*$ for $i = 0, 1, \ldots, \left\lfloor \frac{d}{2} \right\rfloor - 1$ (Hibi)
- $h_0^* + h_1^* + \cdots + h_i^* \leq h_s^* + h_{s-1}^* + \cdots + h_{s-i}^*$ for $i = 0, 1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor$ (Stanley)
- if $h_d^* \neq 0$ then $1 \leq h_1^* \leq h_i^*$ for $i = 1, \ldots, d-1$ (Hibi)
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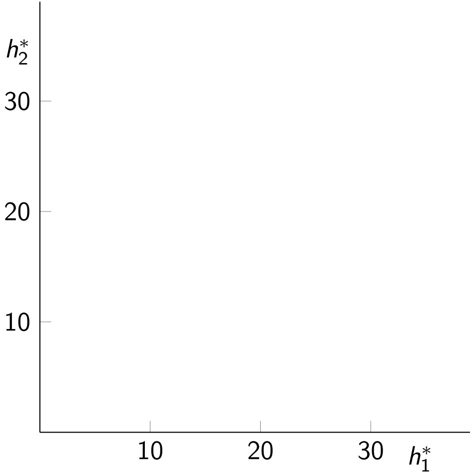
characterize all Ehrhart polynomials $\leftrightarrow$ characterize all h^*-vectors

$$(h_0^*, h_1^*, \ldots, h_{d-1}^*, h_d^*)$$

2-dimensional lattice polytopes



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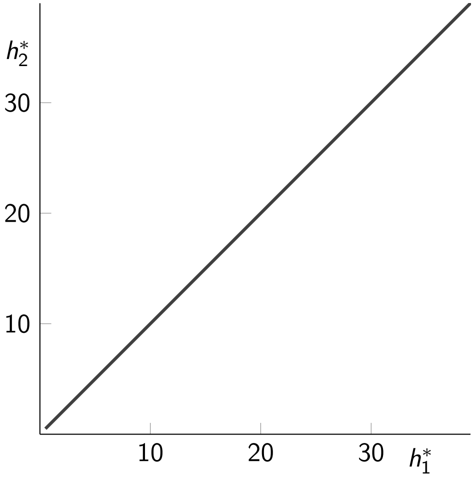

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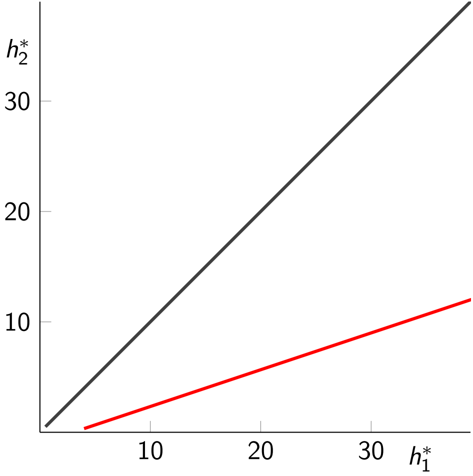
Theorem (Scott, 1976)

$(1, h_1^*, h_2^*)$ is h^*-vector if and only if

- $h_2^* = 0$, *or*
- $h_1^* \leq 3h_2^* + 3$, *or*
- $h^* = (1, 7, 1)$.



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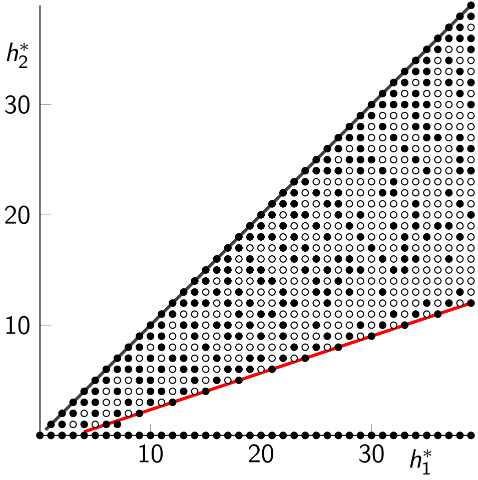

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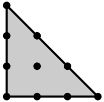
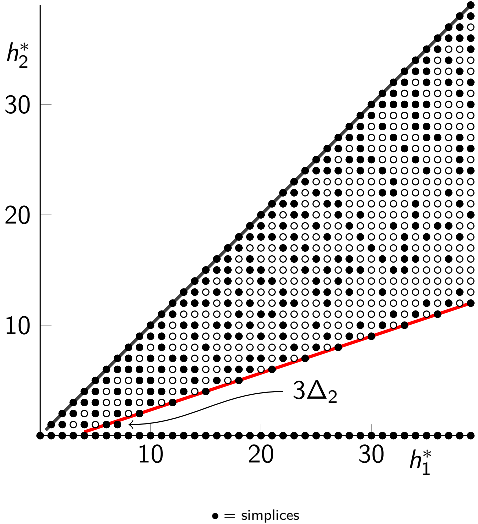
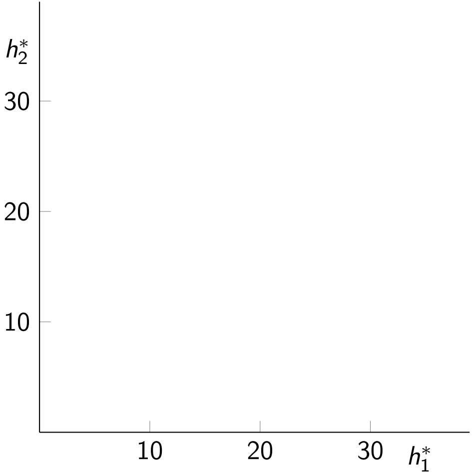
$3\Delta_2 =$ 



3-dimensional hollow lattice polytopes, i.e. no interior points ($h_3^* = 0$)



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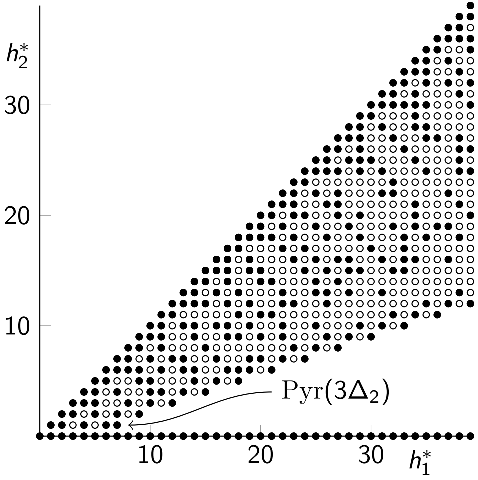

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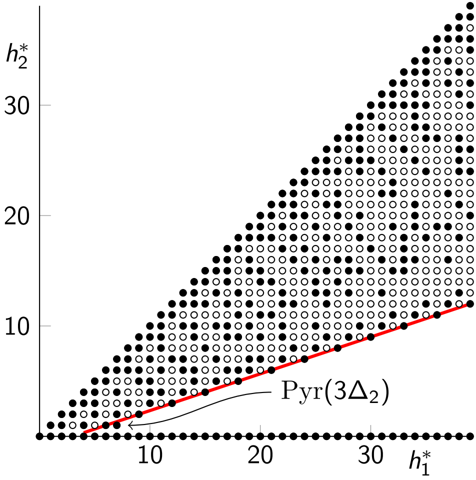

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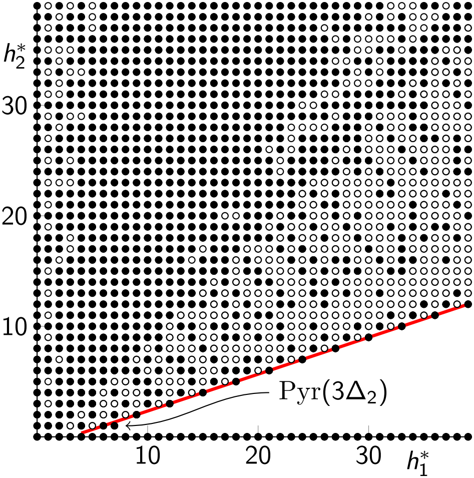
$\Leftarrow$



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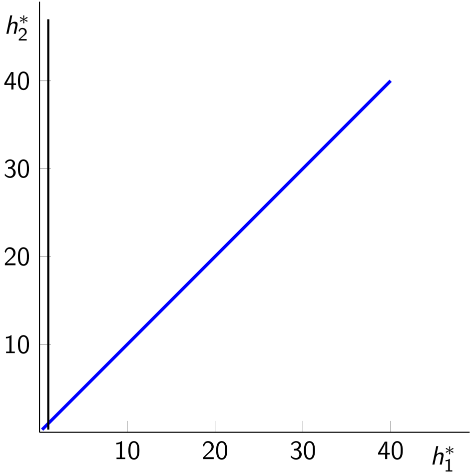
Theorem (Kasprzyk, 2010)

There are 674,688 3-dimensional lattice polytopes with 1 interior lattice point.



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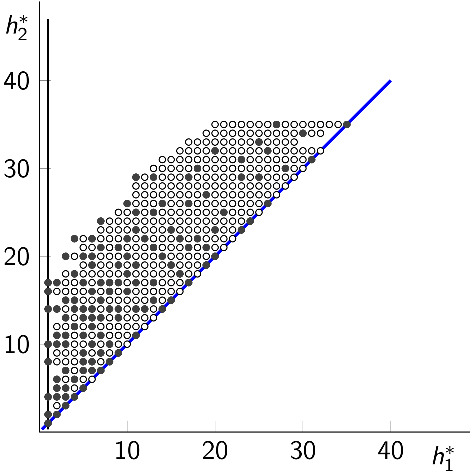

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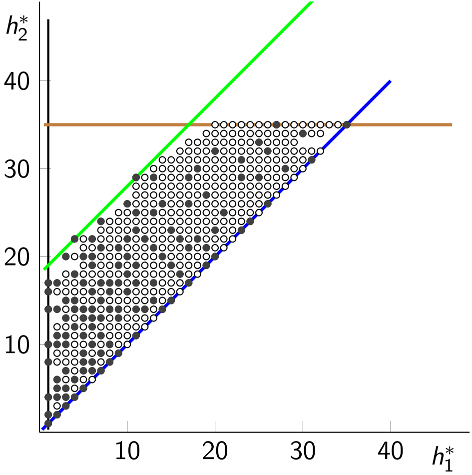

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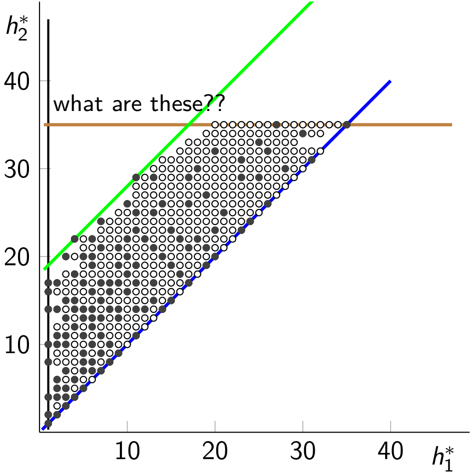

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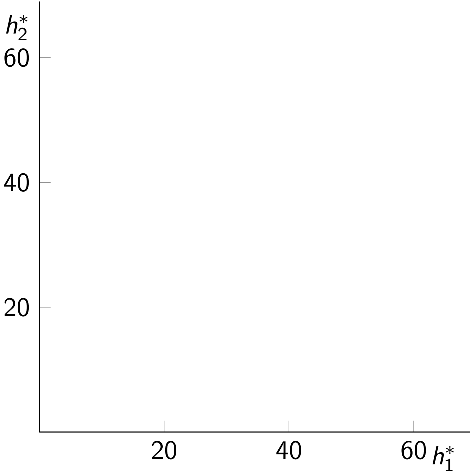
3-dimensional lattice polytopes with 2 interior point ($h_3^* = 2$)



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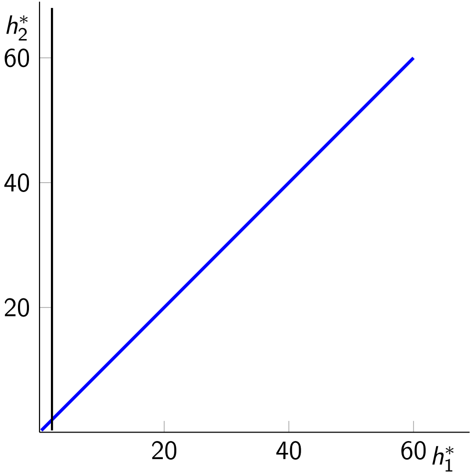

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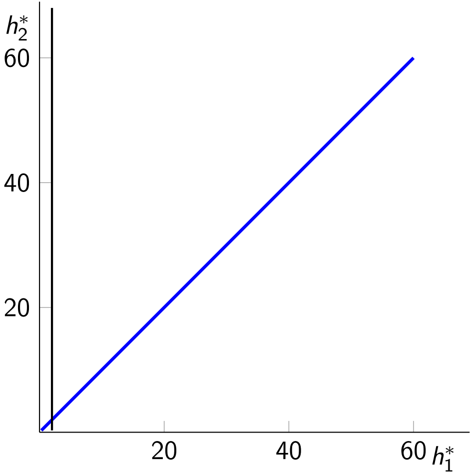
Theorem (B.,Kasprzyk, 2016)

There are 22,673,449 3-dimensional lattice polytopes with 2 interior lattice points.



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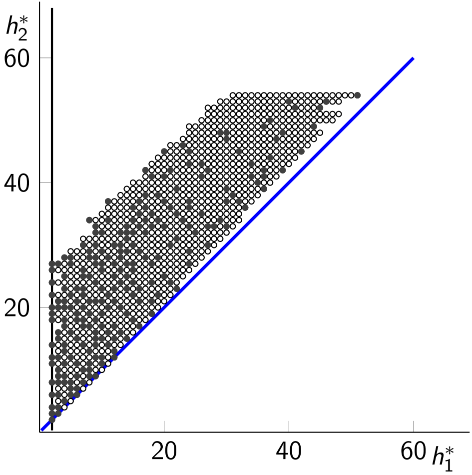

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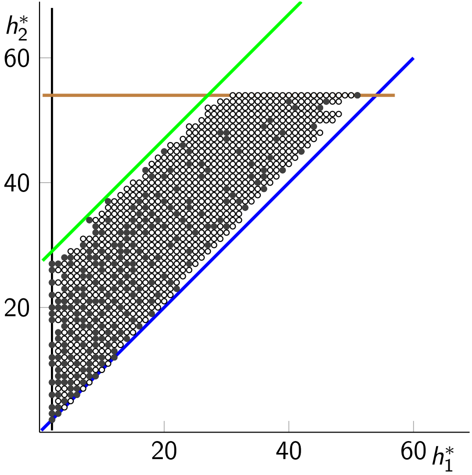

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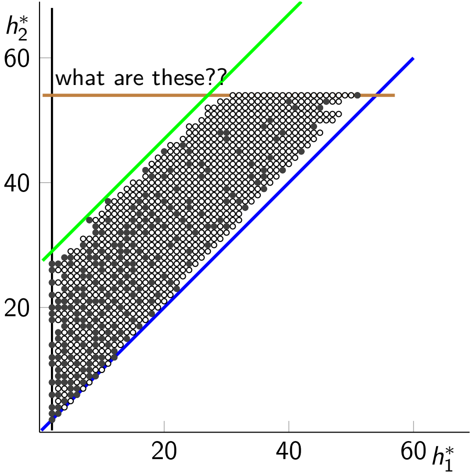
Theorem (B.,Kasprzyk, 2016)

There are 22,673,449 3-dimensional lattice polytopes with 2 interior lattice points.



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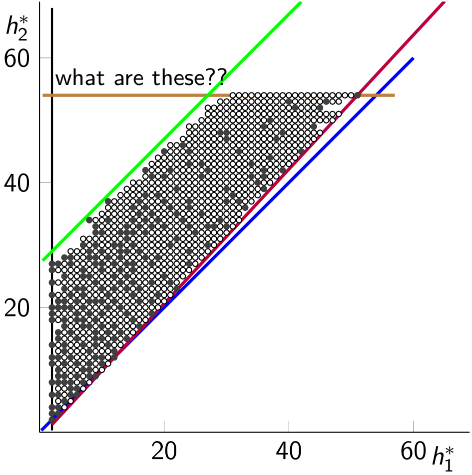

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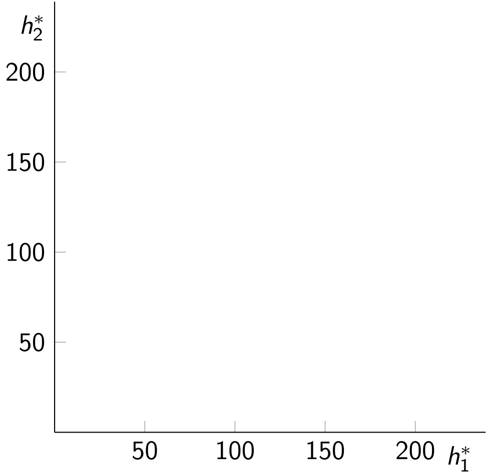

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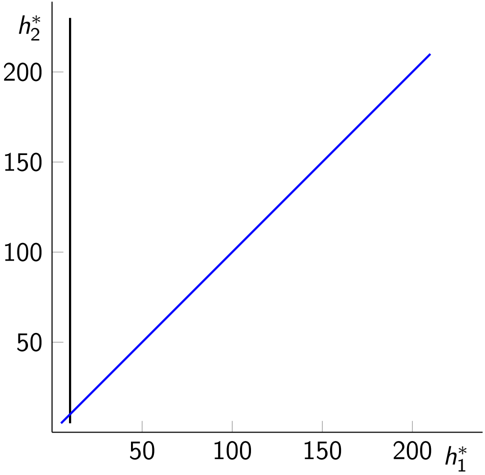

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- $h_2^* \leq 19h_3^* + 16$
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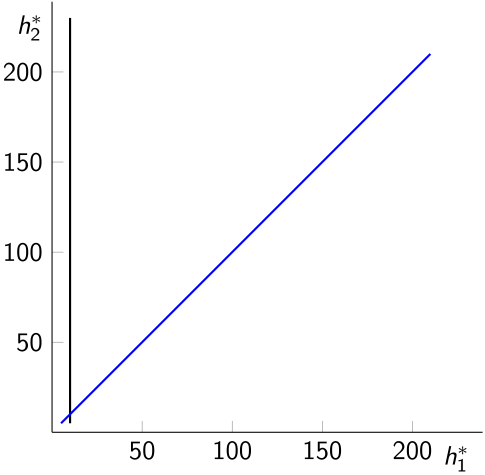

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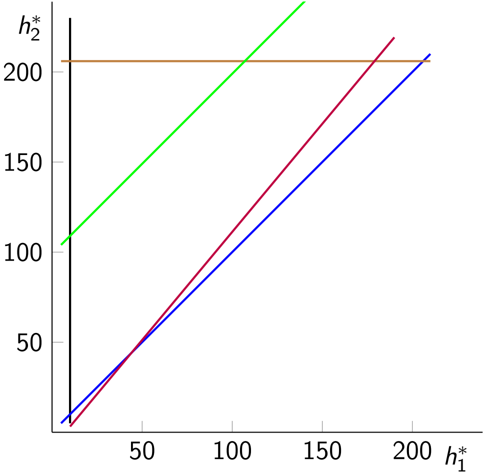

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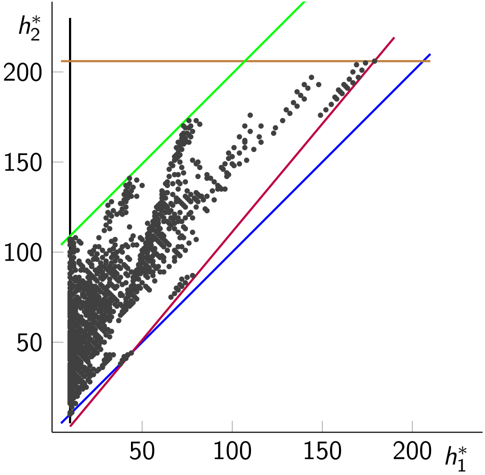

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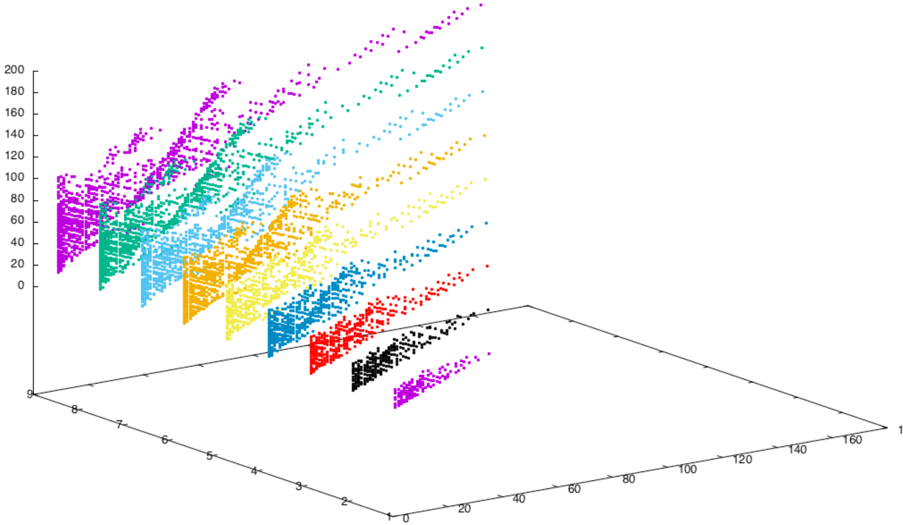
Summary:

There is a lot to be done in Ehrhart Theory, already in dimension 3



A possible approach: the study of the volume

$$\mathrm{Vol}(P) = 1 + h_1^* + \cdots + h_d^*$$

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$$h_1^*, \ldots, h_d^* \quad \rightarrow \quad \mathrm{Vol}(P) \qquad \text{(Ehrhart's Theorem)}$$

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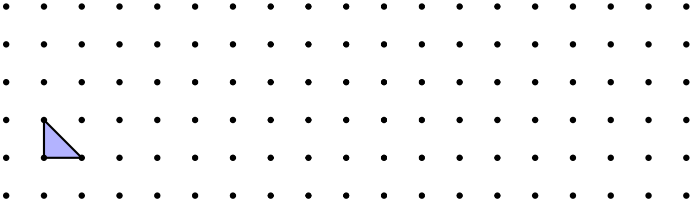

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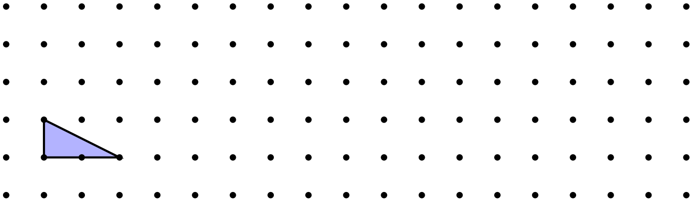

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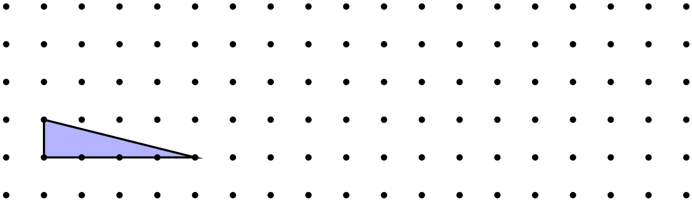

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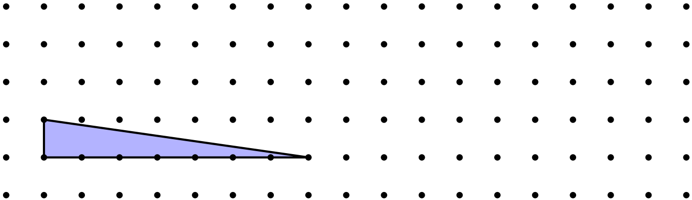

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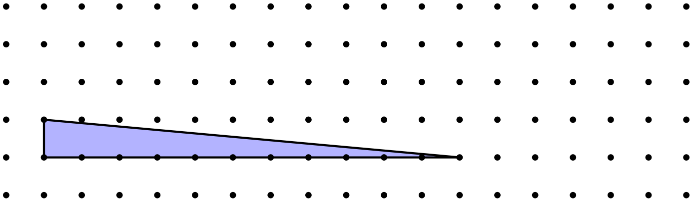

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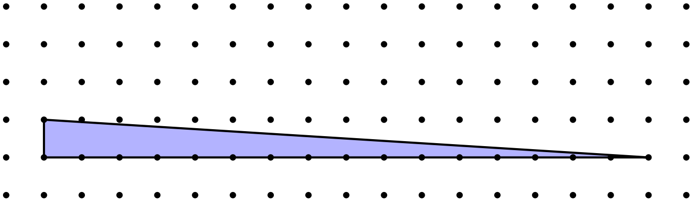

$$1 \leq \mathrm{Vol}(P) < +\infty$$

When $|P^\circ \cap N| > 0$ things get interesting!

[1] up to unimodular equivalence

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Theorem (Lagarias-Ziegler 1991)

A family $\mathcal{F}$ of d-dimensional lattice polytopes is finite [1] *if and only if*

$$\exists\, K \text{ s.t. } \mathrm{Vol}(P) < K \text{ for each } P \in \mathcal{F}$$

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Theorem (Hensley 1983)

P d-dimensional with at least $k > 0$ interior lattice points. Then

$$\mathrm{Vol}(P) < K_{d,k}$$

[1] up to unimodular equivalence

$$Vol(P) < K_{d,h_d^*}$$

How does K_{d,h_d^*} look like?

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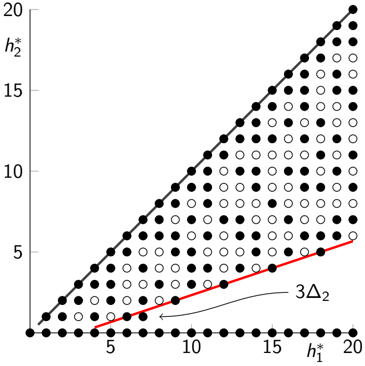
In dimension 2 it's a corollary of Scott's inequality:



$$\mathrm{Vol}(P) \leq 9 \quad \text{if } h_2^* = 1, \qquad \mathrm{Vol}(P) \leq (h_2^* + 1)4 \quad \text{if } h_2^* > 1$$

Corollary: it is possible to list the d-dimensional polytopes with k-interior points!

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d \ h^*_d	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										

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d \ h^*_d	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2										
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- $d = 1 \leftarrow$ trivial

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d \ h_d^*	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	16									
3										
4										

- $d = 1 \leftarrow$ trivial
- $d = 2$
  - $h_d^* = 1 \leftarrow$ Rabinowitz, 1989

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d \ h_d^*	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	16	45								
3										
4										

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d \ h_d^*	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	16	45	120	211	403	714	1,023	1,830	2,700	3,659
3										
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4										

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Corollary: it is possible to list the d-dimensional polytopes with k-interior points!

$d \backslash h^*_d$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	16	45	120	211	403	714	1,023	1,830	2,700	3,659
3	674,688	22,673,449	?	?	?	?	?	?	?	?
4	?	?	?	?	?	?	?	?	?	?

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Nothing else so far

$$Vol(P) < K_{d,h_d^*}$$

How does K_{d,h_d^*} look like?

Many improvements in the last 30 years (Hensley,Lagarias-Ziegler,Pikhurko,...):

$$K_{d,h_d^*} = d!(8d)^d 15^{d2^{2d+1}} h_d^*$$

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Example:
Let's specialize it for $d = 3$ and $h_3^* = 1$. In this case we have a classification and we know that: $\mathrm{Vol}(P) \leq 72$

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We get: $\mathrm{Vol}(P) \leq$ 34500743386148676539022993323077825913472286417970674665589882473299291391966484109283289925714154456681785530177326360513125252460819749279137157726514010336646245182211353482004906943892249369955387872494275400942814293542065143372929389662931047961933795598772230747868553404519189658702145163107754215076205771646472595518186929251060826865950393267615668468592729361015189011695690281260623343131477627765413274119055131450295448303222656250000000000

probably not sharp

For $d \geq 3$ Zaks, Perles and Wilks (1982) described a possible candidate for maximizing the volume (the **ZPW-simplex**)

$$S_k^d := \mathrm{conv}(\mathbf{0}, s_1 e_1, \ldots, s_{d-1} e_{d-1}, (k+1)(s_d - 1) e_d), \qquad \text{where } k \geq 1,$$

has k interior points and very large volume.
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CONJECTURE

$d \geq 3, h_d^* \geq 1$

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Equality only for ZPW-simplices (one extra non-ZPW-simplex for $d = 3$, $h_d^* = 1$).

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Proved in some special cases

- when P is a simplex and $h_d^* = 1$ (Averkov-Krümpelmann-Nill 2015)
- when $d = 3$, $h_d^* = 1$ (Kasprzyk 2010)
- when $d = 3$, $h_d^* = 2$ (B.-Kasprzyk 2016)
- when P is a reflexive polytope (B.-Kasprzyk-Nill 2016)

those are interesting cases, as they contain some ZPW-simplices!

Suppose $\mathbf{0} \in P^\circ$, we define the **dual** of P as

$$P^* := \{y \in (\mathbb{R}^d)^* | \langle y, x \rangle \geq -1 \text{ for every } x \in P\}$$

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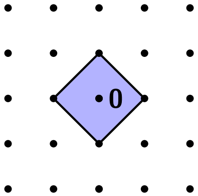

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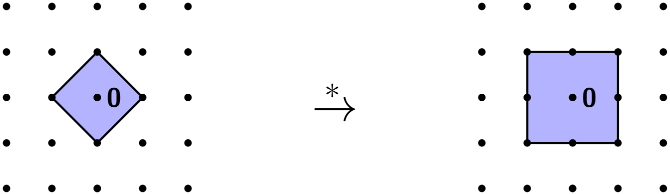

N lattice M lattice

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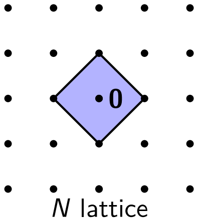
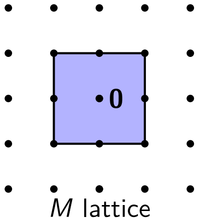


$\xrightarrow{*}$



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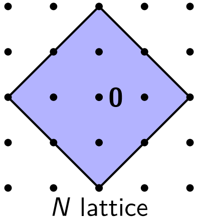

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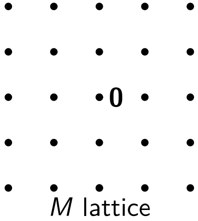

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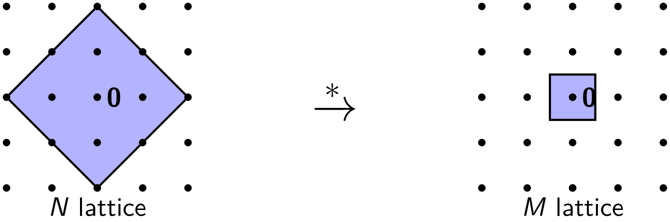

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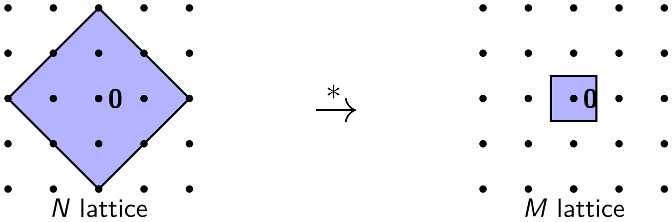

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Definition

If P^* is a lattice polytope, we call P **reflexive**.

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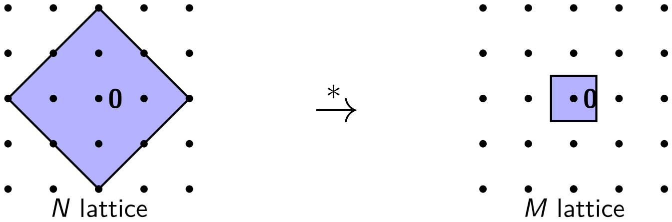
$$P^* := \{y \in (\mathbb{R}^d)^* | \langle y, x\rangle \geq -1 \text{ for every } x \in P\}$$



$P^* \cap M = \{\mathbf{0}\} \Rightarrow P^* \cap M = \{\mathbf{0}\}$, but P^* may not be a lattice polytope!

Definition

If P^* is a lattice polytope, we call P **reflexive**.

- $(P^*)^* = P$ (so P reflexive iff P^* reflexive)
- P reflexive $\Rightarrow$ P has one interior point
- $P \subset Q \Rightarrow P^* \supset Q^*$

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- The ZPW-simplices S_1^d are reflexive

A dual version of the conjecture is valid for $h^*_d = 1$.

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Let $d \geq 3$. If P is a lattice polytope with $h^*_d = 1$ then

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Corollary 2
Let X be a d-dimensional toric Fano variety with at worst canonical singularities, where $d \geq 3$. Then its anticanonical degree is bounded by

$$(-K_X)^d \leq 2(s_d - 1)^2$$

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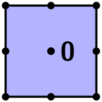

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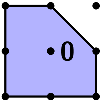

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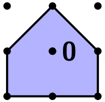

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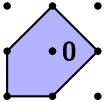

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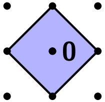

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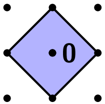

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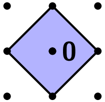

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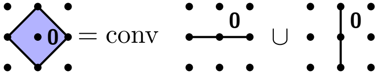
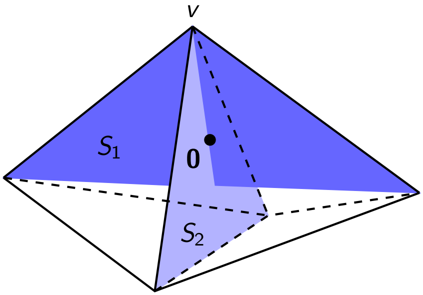


Strategy for the proof

In higher dimensions the decomposition

$$P = \mathrm{conv}(S_1 \cup \cdots \cup S_t)$$

may have a more complex structure



Strategy for the proof

A consequence of the decomposition of P in simplices is

$$P^* \subseteq S_1^* \times \cdots \times S_t^*.$$

We use the **monotonicity of the normalised volume** $\mathrm{Vol}(P) := d!\mathrm{vol}(P)$:

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and the sharp bound for the volume of the dual of simplices with one interior lattice point (AKN)

$$\begin{aligned}\mathrm{Vol}(P^*) &\leq \mathrm{Vol}(S_1^* \times \cdots \times S_t^*)\\ &= (d_1 + \cdots + d_t)! \prod_{i=1}^{t} \mathrm{vol}(S_i^*)\\ &\leq (d_1 + \cdots + d_t)! \prod_{i=1}^{t} \frac{1}{d!}\, 2(s_{d_i} - 1)^2\end{aligned}$$

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\end{aligned}$$

We are left with

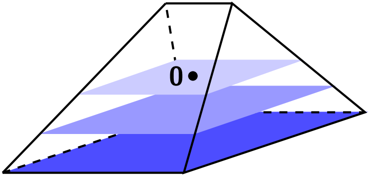
- the case in which the minimal polytope P decomposes in two $(d-1)$-dimensional simplices
- a few cases in dimension up to 5.

Case 1: P decomposes in two $(d-1)$-dimensional simplices

Strategy for the proof

Case 1: P decomposes in two $(d-1)$-dimensional simplices

- describe P^* as **union of slices**.



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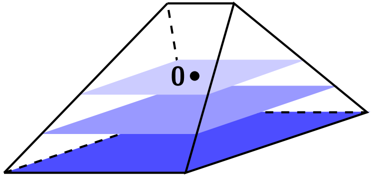

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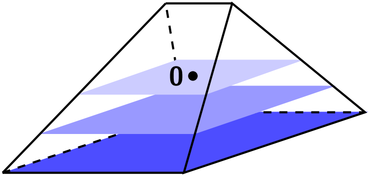

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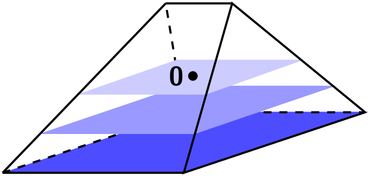

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We express $\mathrm{vol}(P^*)$ in terms of barycentric coordinates of the S_i and we use existing bounds (AKN)

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Strategy for the proof

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- P^* decomposes in two lower dimensional simplices.

  integration technique $+$ classification of barycentric coordinates in dimensions up to 4

  check that the bound holds for all of them

Case 2: finitely many cases in dimension ≤ 5

- P^* decomposes in two lower dimensional simplices.

  integration technique + classification of barycentric coordinates in dimensions up to 4

  check that the bound holds for all of them
- P^* decomposes in more lower dimensional simplices.

  decomposition + classification of simplices with $h^*_d = 1$ of dimension up to 3

  build minimal polytopes and check that the bound holds for all of them

What's next? Fix $d \geq 3$ and $h_d^* > 0$.

It is clear that

$$\left|P \cap \mathbb{Z}^d\right| < J_{d,h_d^*}$$

Not much known. Not even a candidate for J_{d,h_d^*}.

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Conjecture (BK 16)

For $d = 3$ and $h_3^* > 1$

$$h_1^* \leq 16h_3^* + 19 \qquad \text{and} \qquad h_2^* \leq 19h_3^* + 16.$$

Thank you!