A characterization of simplicial manifolds with $g_2 \leq 2$

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January 17, 2017

Outline

- Basics on simplicial complexes and known theorems.
- Three different retriangulations of complexes.
- Main theorems.
- Additional remarks and open problems.

Simplicial complexes

Definition

A *simplicial complex* Δ on vertex set V is a collection of subsets $\tau \subseteq V$, called faces, that is closed under inclusion.

For a simplicial complex Δ, define:

1. $\dim \tau := |\tau| - 1$ for $\tau \in \Delta$,;
2. $\dim \Delta := \max\{\dim \tau : \tau \in \Delta\}$;
3. a *facet* τ is a maximal face under inclusion;
4. the *star* of a face τ is $\mathrm{st}_\Delta \tau := \{\sigma \in \Delta : \sigma \cup \tau \in \Delta\}$;
5. the *link* of a face τ is $\mathrm{lk}_\Delta \tau := \{\sigma - \tau \in \Delta : \tau \subseteq \sigma \in \Delta\}$;
6. a *missing face* τ is a subset of $V(\Delta)$ such that τ is not a face of Δ but every proper subset of τ is.

Δ is called *pure*, if all of its facets have the same dimension.

Δ is called *prime*, if it is pure and has no missing facets.

Face-number related invariants

Let Δ be a $(d-1)$-dimensional simplicial complex.

Definition

The f-number $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. The vector $(f_{-1}, f_0, \cdots, f_{d-1})$ is called the f-vector.

Definition

The h-vector of Δ, $(h_0, h_1, \cdots, h_d)$, is defined by the relation $\sum_{j=0}^{d} f_{j-1}(x-1)^{d-j} = \sum_{i=0}^{d} h_i x^{d-i}$.

Definition

The g-vector of Δ is $(g_0, g_1, \cdots, g_{\lfloor \frac{d}{2} \rfloor})$ whose entries are given by

1. $g_0 = 1$;
2. $g_i = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Main object

For any simplicial complex Δ, there is an associated topological space $||\Delta||$.

1. A *polytope* is the convex hull of a finite set of points in some $\mathbb{R}^e$. It is called a *simplicial* d-polytope if it is d-dimensional and all of its facets are simplicial. The boundary complex of a simplicial polytope is called a *polytopal sphere*.
2. A *simplicial sphere* (resp. *manifold*) is a simplicial complex whose geometric realization is homeomorphic to a sphere (resp. manifold).

Central question: Characterize simplicial manifolds with small g_i, $i \geq 2$.

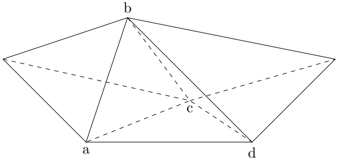
$(r-1)$-stacked spheres

Given a simplicial ball Δ, the faces of $\Delta - \partial\Delta$ are called the *interior* faces of Δ.

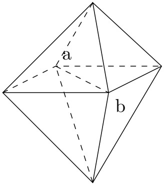
Definition

Let Δ be a simplicial d-ball. Δ is said to be $(r-1)$-*stacked* if Δ has no interior k-faces for $k \leq d-r$. An $(r-1)$-*stacked* simplicial sphere is the boundary complex of an $(r-1)$-stacked triangulation of a simplicial ball.

Remark: 1-stacked $=$ stacked.



1-stacked



2-stacked

Previous results, $g_2 = 0$

Theorem (Kalai, 1987)

Let Δ be a simplicial manifold of dimension $d \geq 2$. Then $g_2 \geq 0$. Furthermore, if $d \geq 3$, then equality holds if and only if Δ is a stacked sphere.

Remarks:

1. The theorem continues to hold in the class of normal pseudomanifolds.
2. The proof is based on rigidity theory of graphs.

Previous results, $g_i = 0$ for $i \geq 2$

Theorem (Murai-Nevo, 2013)

Let Δ be polytopal $(d-1)$-sphere and $2 \leq r \leq d/2$. Then $g_r(\Delta) = 0$ if and only if Δ is $(r-1)$-stacked.

Remarks:

1. The theorem is also true for homology spheres with weak Lefschetz property over a field of characteristic 0.
2. The proof is based on commutative algebra tools.

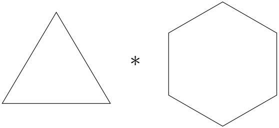
Previous result 3, $g_2 = 1$

Given two simplicial complexes Δ_1 and Δ_2 on disjoint vertex sets, define their join, $\Delta_1 * \Delta_2$, as the complex on vertex set $V(\Delta_1) \cup V(\Delta_2)$ whose faces are $\{\tau_1 \cup \tau_2 : \tau_1 \in \Delta_1, \tau_2 \in \Delta_2\}$. We also denote the boundary complex of the i-dimensional simplex as $\partial\sigma^i$.

Theorem (Nevo-Novinsky, 2011)

Let $d \geq 4$, and let Δ be a prime simplicial $(d-1)$-sphere and $g_2(\Delta) = 1$. Then Δ is combinatorially isomorphic to one of the following cases:

1. $\partial\sigma^i * \partial\sigma^{d-i}$, *where* $2 \leq i \leq d-2$;
2. $\partial\sigma^{d-2} * C$, *where C is a cycle.*



Central retriangulation, definition

Definition

Assume that

- Δ is a d-dimensional simplicial complex;
- B is a subcomplex of Δ which is also a simplicial d-ball.

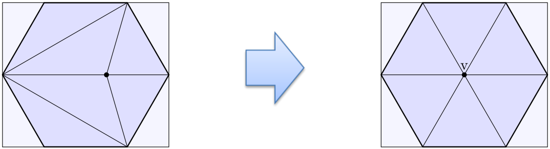
The **central retriangulation** of Δ along B, denoted as $\mathrm{crtr}_B(\Delta)$, is the new complex we obtain after

1. removing all of the interior faces of B;
2. replacing them with the interior faces of the cone on ∂B.

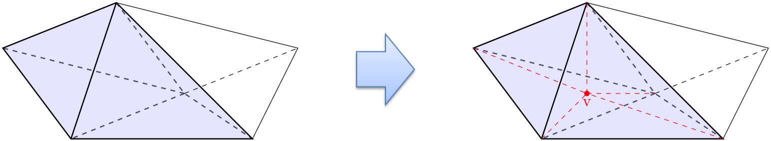
(the cone point is a new vertex v.)

Central retriangulation, examples

Examples:



$$f(\Delta) = f(\mathrm{crtr}_B(\Delta)) = (1, 11, 20, 10)$$



$$g_2(\mathrm{crtr}_B(\Delta)) = g_2(\Delta) + \Big(\#\text{ of new edges} - 4 \cdot (\#\text{ of new vertices}) = 1\Big).$$

Central retriangulation, properties

Any $(r-1)$-stacked sphere S uniquely determines an $(r-1)$-stacked ball whose boundary complex is S. We denotes this $(r-1)$-stacked ball by $S(r-1)$.

Properties:

1. If the retriangulated subcomplex B is an $(r-1)$-stacked ball, where $2 \leq r \leq d/2$, then $g_i(\mathrm{crtr}_B(\Delta)) = g_i(\Delta) + g_{i-1}(\partial B)$ for $1 \leq i \leq d/2$.
2. If B is the union of some facets of Δ whose facet-ridge graph is a tree, then B is a stacked ball.

Inverse stellar retriangulation, definition

Definition

Let Δ be a d-dimensional simplicial complex. Assume that there is a vertex $v \in V(\Delta)$ such that

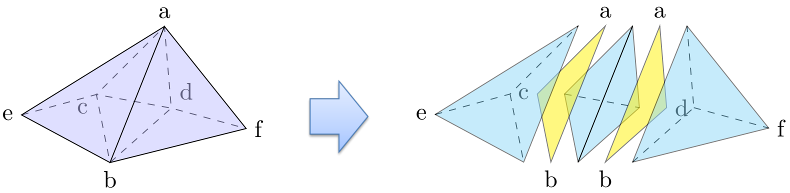
1. $\mathrm{lk}_\Delta\, v$ is a $(r-1)$-stacked $(d-1)$-sphere for some $2 \leq r \leq d/2$;
2. no interior face of $(\mathrm{lk}_\Delta\, v)(r-1)$ is a face of Δ.

Then define the *inverse stellar retriangulation* on vertex v by

$$\mathrm{sd}_v^{-1}(\Delta) = (\Delta \backslash \{v\}) \cup (\mathrm{lk}_\Delta\, v)(r-1).$$

Inverse stellar retriangulation, properties

Example:



$\mathrm{lk}_\Delta\, v$ union of three 3-simplices

Properties of inverse stellar retriangulation:

1. $\mathrm{sd}_v^{-1}(\Delta)$ is PL-homeomorphic to Δ.
2. $g_i(\mathrm{sd}_v^{-1}(\Delta)) = g_i(\Delta) - g_{i-1}(\mathrm{lk}_\Delta\, v)$ for $1 \leq i \leq d/2$.

The Swartz operation, part 1

Definition

Let Δ be a simplicial $(d-1)$-manifold. If a missing facet τ of $\mathrm{lk}_\Delta v$ is also a missing face of Δ, then we define the *Swartz operation* on (v, τ) of Δ by

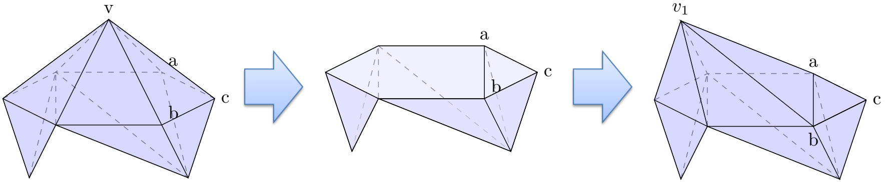
1. removing v;
2. adding τ to Δ;
3. coning off two remaining simplicial spheres S_1, S_2 with two new vertices v_1, v_2.

(Here S_1, S_2 are the two simplicial spheres such that their connected sum by identifying the face τ is $\mathrm{lk}_\Delta v$.)

The Swartz operation, part 2

1. If any of the two spheres, say S_1, is the boundary complex of a simplex, then we add the simplex to $\Delta \cup \{\tau\}$ instead of coning off S_1 with v_1. The resulting complex is denoted by $\mathrm{so}_{v,\tau}(\Delta)$.
2. If $\dim \Delta \geq 3$, then iterating this process, we add all missing facets of $\mathrm{lk}_\Delta v$ to Δ. The resulting complex is denoted by $\mathrm{so}_v(\Delta)$.

Example:



The Swartz Operation, properties

Properties of the Swartz Operation:

1. $\mathrm{so}_v^{-1}(\Delta)$ is PL-homeomorphic to Δ.
2. If $\mathrm{lk}_\Delta v$ is stacked, then $\mathrm{so}_v(\Delta) = \mathrm{sd}_v^{-1}(\Delta)$.
3. If Δ is of dimension ≥ 3, then

$$g_2(\mathrm{so}_v(\Delta)) = g_2(\Delta) - \#\{\text{missing facets of } \mathrm{lk}_\Delta v\}.$$

Main Theorem 1

Simplification: It suffices to consider *prime* simplicial manifolds.

Theorem (Zheng, 2016)

Let $d \geq 5$, and let Δ be a prime simplicial $(d-1)$-manifold with $g_2(\Delta) = 1$. Then Δ is obtained by centrally retriangulating a stacked $(d-1)$-sphere along the star of one of its faces, whose dimension, i, satisfies $0 < i < d-1$.

Corollary

This Δ is either the join of $\partial\sigma^i$ and $\partial\sigma^{d-i}$, where $2 \leq i \leq d-2$, or the join of $\partial\sigma^{d-2}$ and a cycle.

Sketch of Proof, part 1

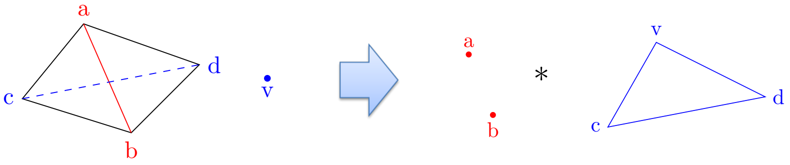
The starting point:

$$\sum_{v \in V(\Delta)} g_2(\mathrm{lk}_\Delta\, v) = d - 1 + 3g_3(\Delta) = d + 2 \text{ or } d - 1.$$

Case 1: Every vertex link has $g_2 = 1$.

$\Rightarrow$ Δ is the join of two boundary complexes of simplices.

$\Rightarrow$ It is obtained by centrally retriangulating $\partial\sigma^d$ along the star of a face.

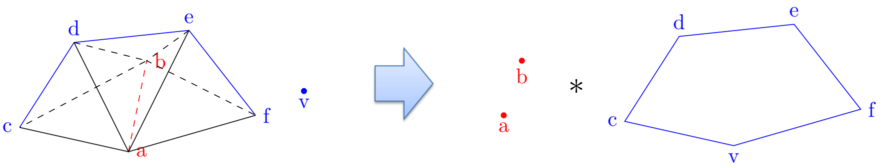


Sketch of proof, part 2

Case 2: There is a vertex link with $g_2 = 0$.

$\Rightarrow$ Δ is the join of a cycle and $\partial\sigma^{d-2}$.

$\Rightarrow$ It is obtained by centrally retriangulating a certain stacked sphere along the union of two facets.



Main Theorem 2

Theorem (Zheng, 2016)

Let $d \geq 4$. Every prime simplicial $(d-1)$-manifold with $g_2 = 2$ is either the octahedral 3-sphere (in this case $d = 4$), or it can be obtained from a polytopal $(d-1)$-sphere with $g_2 = 0$ or 1, by centrally retriangulating along some stacked subcomplex.

Corollary

All simplicial $(d-1)$-spheres with $g_2 \leq 2$ are polytopal.

Additional remarks

1. Both of the main theorems continue to hold for the class of normal pseudomanifolds when the dimension $d-1$ is at least 4.
2. For all $d \geq 4$, there are triangulations of $\mathbb{RP}^2 * \mathbb{S}^{d-4}$ that have $g_2 = 3$. This implies that our theorem cannot be extended to higher g_2 in the class of normal pseudomanifolds.
3. It appears that many non-polytopal spheres that are known so far have $g_2 \geq 5$:
   the Barnette sphere: $g_2 = 5$;
   all non-polytopal 3-spheres with nine vertices: $g_2 \geq 5$.

Open problems

1. What is the minimum value of g_2 for non-polytopal $(d-1)$-spheres?
2. Characterize simplicial spheres with $g_i = 1$ for any $i \geq 3$.
3. Characterize simplicial balls with $g_2(\Delta, \partial\Delta) = 1$.

Thank You!