The Finiteness Threshold Width of Lattice Polytopes

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All of this is joint work with

Mónica

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Paco

The finiteness threshold width of lattice polytopes
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Classifying lattice polytopes

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Definition: Equivalent polytopes

Two lattice polytopes P and P' are called *unimodularly equivalent*, if there is a lattice-preserving affine isomorphism mapping them onto each other, i.e. $P' = AP + b$, with $A \in GL_d(\mathbb{Z}), b \in \mathbb{Z}^d$.

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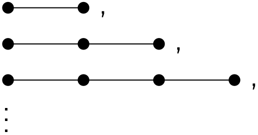
- Finitely many lattice polytopes := finitely many up to the above equivalence relation

Classifying lattice polytopes

Naive approach: Enumerate by dimension and size ($|P \cap \mathbb{Z}^d|$) of P

- $d = 1$ works great:

$\lvert P \cap \mathbb{Z}^d \rvert$	# different polytopes
2	1
3	1
4	1
$\vdots$	$\vdots$



Classifying lattice polytopes

Naive approach: Enumerate by dimension and size ($n := |P \cap \mathbb{Z}^d|$) of P

- $d = 2$ also works:

n	# different polytopes
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5	6
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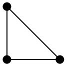
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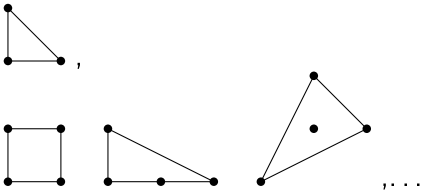

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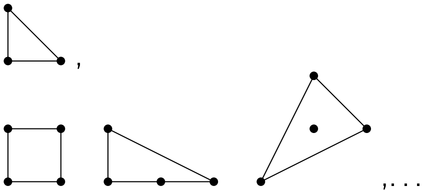

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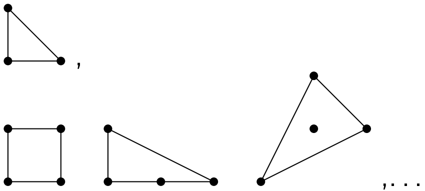

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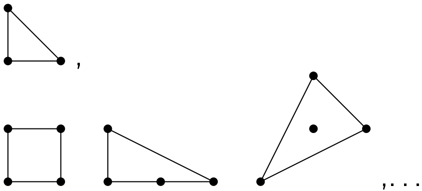

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- Pick's Theorem, $\mathrm{vol}(P) = 2i + b - 2 \implies \mathrm{vol}(P) \leq 2n - 5$
- Every full-dimensional polytope contains a standard simplex.

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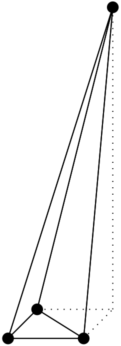

Reeve tetrahedra:

$$T_r := \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & r \end{pmatrix}$$

$$\operatorname{vol}(T_r) = r \implies \begin{array}{c|c} n & \#\ P \\ \hline 4 & \infty \end{array}$$

Rescue in dimension 3

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Definition: (lattice) width of a polytope

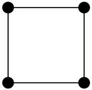
The *width* of a lattice polytope P *with respect to a linear functional* $\ell \in (\mathbb{R}^d)^*$ is defined as

$$\mathrm{width}_\ell(P) := \max_{p,q \in P} |\ell \cdot p - \ell \cdot q| \ ,$$

and the *(lattice) width* of P is the minimum such $\mathrm{width}_\ell(P)$ where ℓ ranges over non-zero integer functionals:

$$\mathrm{width}(P) := \min_{\ell \in (\mathbb{Z}^d)^* \setminus \{0\}} \mathrm{width}_\ell(P).$$

Width examples



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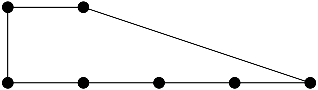

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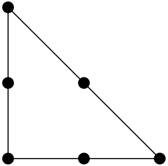
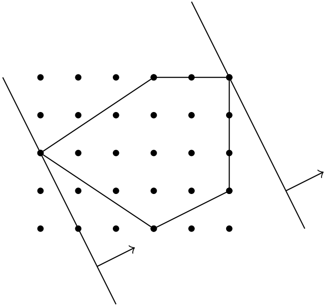


$\text{width}(P) = 2$

Width quiz



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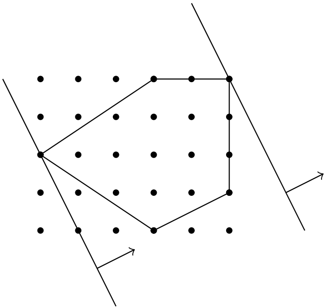

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- For more complicated examples, use polymake ($P->LATTICE_WIDTH).

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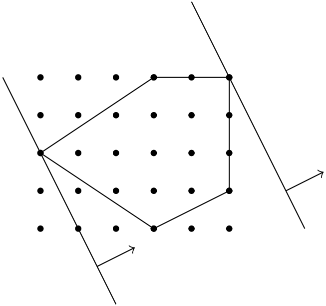

width(P) = 4

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Enumerating lattice 3-polytopes

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# **vertices**	4	5	6	7	8	9	10	total
size 5	9	0	–	–	–	–	–	9
size 6	36	40	0	–	–	–	–	76
size 7	103	296	97	0	–	–	–	496
size 8	193	1195	1140	147	0	–	–	2675
size 9	282	2853	5920	2491	152	0	–	11698
size 10	478	5985	18505	16384	3575	108	0	45035
size 11	619	11432	48103	64256	28570	3425	59	156464

Table: Lattice 3-polytopes of width larger than one and size ≤ 11, classified according to their size and number of vertices.

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- How about the general case?

The finiteness threshold width

Definition: Finiteness threshold width

For each d and each $n \geq d+1$, denote by $w^{\infty}(d,n) \in \mathbb{N} \cup \{\infty\}$ the minimal width $W \geq 0$ such that there exist only finitely many lattice d-polytopes of size n and width $> W$.
Let $w^{\infty}(d) := \sup_{n \in \mathbb{N}} w^{\infty}(d,n)$.

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- $w^{\infty}(3) = 1$

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Definitions: Hollow polytopes, empty polytopes, $w_H(d)$ & $w_E(d)$

We say a lattice polytope is *hollow*, if $P \cap \mathbb{Z}^d \subseteq \delta P$. We say it is *empty*, if $P \cap \mathbb{Z}^d = \mathrm{vert}(P)$.
We denote by $w_H(d)$ and $w_E(d)$ the maximum widths of hollow and empty lattice d-polytopes, respectively.

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There are only finitely many empty 4-simplices of width larger than two.

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- Full classification of those polytopes was computed recently by Óscar Iglesias, verifying a conjecture by Christian Haase[H00]

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- $w^\infty(d) \leq w_H(d-1) \leq O(d^{\frac{3}{2}})$ by the flatness theorem [KL88]

Known values

d	$w_E(d-1)$	$w_H(d-2)$	$w^\infty(d)$	$w_H(d-1)$
1	–	–	0	–
2	1	–	0	1
3	1	1	1	2
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5	≥ 4	3	≥ 4	≥ 4

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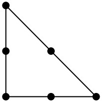
- $w_H(2) = 2$ 

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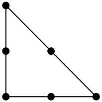
- $w_H(2) = 2$ 
- $w_H(3) = 3$ [AWW11,AKW15]

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$\Longrightarrow w^{\infty}(d) \leq w_H(d-1)$

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Definition: Lift of a polytope

We say that a (lattice) polytope P is a *lift* of a (lattice) $(d-1)$-polytope Q if there is a (lattice) projection π with $\pi(P) = Q$. Without loss of generality, we will typically assume $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ to be the map that forgets the last coordinate.
Two lifts $\pi_1 : P_1 \to Q$ and $\pi_2 : P_2 \to Q$ are *equivalent* if there is a unimodular equivalence $f : P_1 \to P_2$ with $\pi_2 \circ f = \pi_1$.

Lifts of bounded size

Theorem

Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope of width W. Then all lifts $P \subset \mathbb{R}^d$ of Q have width $\leq W$. All but finitely many of them have width $= W$.

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Theorem

For all $d \geq 3$, $w^\infty(d)$ equals the maximum width of a lattice $(d-1)$-polytope Q that admits infinitely many lifts of bounded size. Moreover, Q is hollow.

The case $d = 4$

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- Compute their width with polymake (our algorithm is in version 3.0.)
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- All their subpolytopes have width at most 2.

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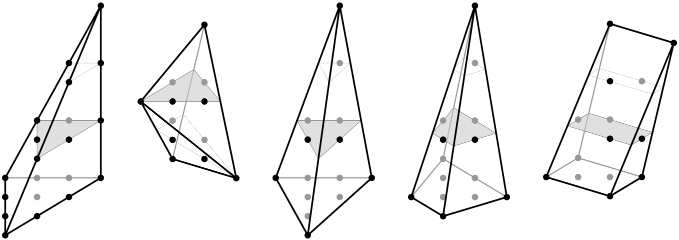

Figure: The five (maximal) hollow 3-polytopes of width three

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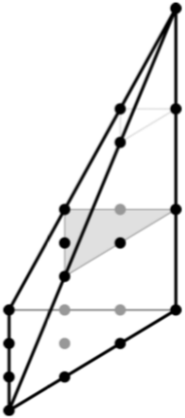

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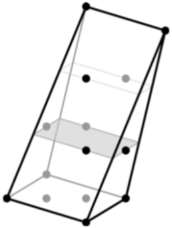

$$\mathcal{M}_{10}\begin{pmatrix} 1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix}$$

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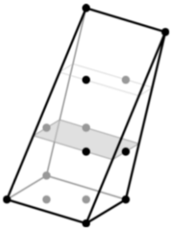

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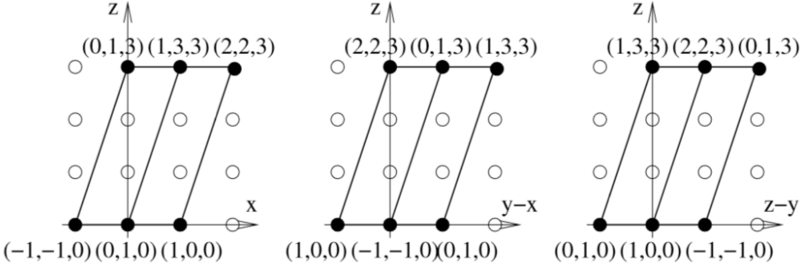

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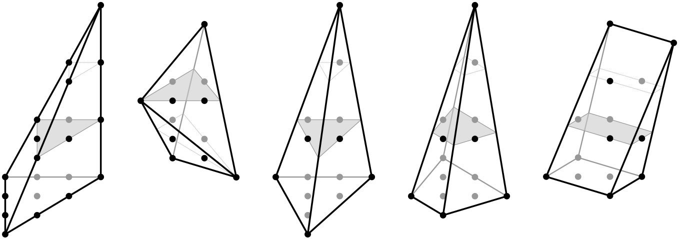

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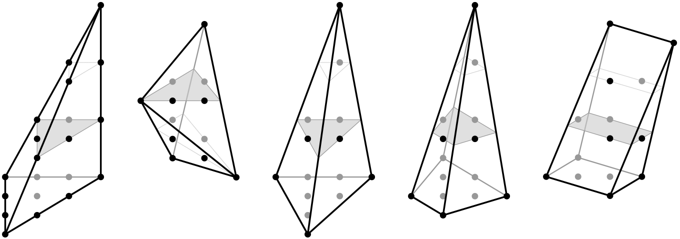

Figure: The five hollow 3-polytopes of width three

To show: These polytopes have only f.m. lifts of bounded size.

Polytopes with finitely many lifts of bounded size

Lemma

Let Q be a lattice pyramid with basis F and apex v. If F has finitely many lifts of bounded size, then so does Q.

Proof:

- It is enough to look at tight lifts, where a lift is called tight if there is a bijection between the vertices of P and Q.

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- Any tight lift of Q is of the form $P(\tilde{F}, h) := \mathrm{conv}(\tilde{F} \cup \{\tilde{v}\})$, where $\tilde{F}$ is a tight lift of F and $\tilde{v} = (v, h)$ is a point in the fiber of v.

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- Let m be the distance of v to F.
- $P(\tilde{F}, h)$ is equivalent to $P(\tilde{F}, h + m)$ for all $h \in \mathbb{Z}$.
- Hence there are at most $m - 1$ values of h that give non-equivalent tight liftings $P(\tilde{F}, h)$, for any fixed $\tilde{F}$.

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Lattice simplices have finitely many lifts of bounded size.

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- True by an inductive argument, as 1-simplices have only finitely many lifts.

The case $d = 4$

Theorem

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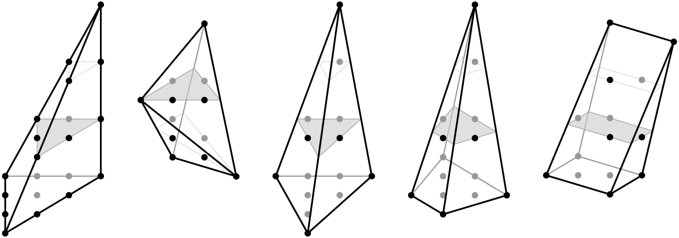

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Open problems

Theorem

1. $w^\infty(d, n) \le w^\infty(d, n+1)$ *for all* d, n.
2. $w^\infty(d) \le w^\infty(d+1)$ *for all* d.

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Questions

- Is $w^\infty(d,n) \leq w^\infty(d+1,n+1)$? (True for $n = d+1$ [HZ00])
- Is $w^\infty(d) = w^\infty(d,d+1)$? (True for $d \leq 4$)

All of this is joint work with

Mónica

Christian

Paco

The finiteness threshold width of lattice polytopes
arXiv:1607.00798