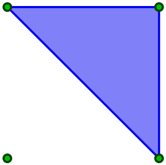
Generalized $(0, 1, 2)$-polytopes

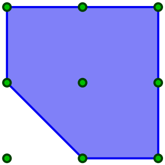
Katy Beeler

Berlin Mathematical School
Berlin, Germany
katybeeler@mi.fu-berlin.de

January 19, 2017

Motivation





Definition

We say a polytope $P \subset \mathbb{R}^d$ is a $(0,1,2)$-**polytope** if for every $v \in \text{vert}(P)$, $v \in \{0,1,2\}^d$.

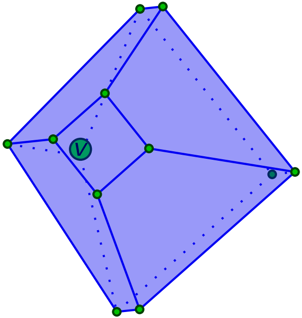
Definition

We say a polytope $P \subset \mathbb{R}^d$ is a $(0, 1, a)$**-polytope** if for every $v \in \mathrm{vert}(P)$, $v \in \{0, 1, a\}^d$.

Example

Consider the following $(0,1,3)$-polytope with vertex set:

$$\left\{ \begin{array}{c} (1,1,0),(3,0,0),(0,3,0),(3,3,0),(0,0,1),(3,0,1), \\ (0,3,1),(3,3,1),(0,0,3),(1,0,3),(0,1,3),(1,1,3) \end{array} \right\}.$$



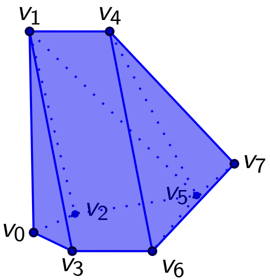
Definition

Let $\mathbf{a} \in R^d$. We say a polytope $P \subset \mathbb{R}^d$ is a $(0,1,a_i)$-**polytope** if for every $v \in \mathrm{vert}(P)$, $v \in \{0,1,a_i\}^d$ for $i \in [d]$.

Example

Consider the following $(0,1,a_i)$-polytope, for $\mathbf{a} = (2,3,4)$, with vertex set:

$$\left\{ \begin{array}{l} (0,0,0),(1,0,0),(0,1,0),(1,3,0), \\ (2,1,0),(2,3,1),(0,0,4),(1,1,4) \end{array} \right\}$$



Question

Can we enumerate these guys?

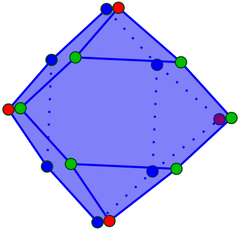
Proposition

A $(0, 1, 2)$-polytope has at most 16 vertices.

Example

Consider the following polytope with vertex set:

$$\left\{ \begin{array}{l} (1,0,0),(2,0,0),(2,1,0),(0,1,0),(0,2,0),(1,2,0), \\ (0,0,1),(2,0,1),(0,2,1),(2,2,1),(0,0,2),(1,0,2), \\ (0,1,2),(2,1,2),(1,2,2),(2,2,2) \end{array} \right\}$$



- $x_3 = 0$
- $x_3 = 1$
- $x_3 = 2$

Question

Which values of *a* allow different combinatorial types?

Find $\det(M) = 0$ for

$$M = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{for } v_i \in \{0,1,a\}^3.$$

Only real solution greater than 2 is $a = 3$.

Find $\det(M) = 0$ for

$$M = \begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{for } v_i \in \{0, 1, a\}^3.$$

Only real solution greater than 2 is $a = 3$.

Different combinatorial types could exist for

$(a) \quad a = 2$	$(b) \quad 2 < a < 3$
$(c) \quad a = 3$	$(d) \quad 3 < a$

Number of combinatorial types of $(0, 1, a)$-polytopes:

no. of vertices	$a = 2$	$2 < a < 3$	$a = 3$	$a > 3$	total
6	7	7	7	7	7
7	34	34	34	34	34
8	193	247	249	249	251
9	680	1215	1415	1406	1462
10	1758	$*$	$*$	$*$	$*$
11	2049	5243	6150	6196	6437
12	955	2814	3134	3153	3319
13	207	663	700	704	745
14	29	90	91	92	97
15	3	10	10	10	10
16	1	2	2	2	2
total	6017	$*$	$*$	$*$	$*$

Question

Can we enumerate $(0, 1, a_i)$-polytopes employing the previous approach?

Can we enumerate employing the previous approach?

Consider the 4×4 matrix

$$M = [m_{ij}]$$

where $m_{ij} \in \{0, 1, a_i\}$ and $m_{4j} = 1$.

Can we enumerate employing the previous approach?

Consider the 4×4 matrix

$$M = [m_{ij}]$$

where $m_{ij} \in \{0, 1, a_i\}$ and $m_{4j} = 1$.

- 334 determinants with solutions in the feasible range, $\mathbb{R}^3_{\geq 2}$

Can we enumerate employing the previous approach?

Consider the 4×4 matrix

$$M = [m_{ij}]$$

where $m_{ij} \in \{0, 1, a_i\}$ and $m_{4j} = 1$.

- 334 determinants with solutions in the feasible range, $\mathbb{R}^3_{\geq 2}$
- Choose $\sim$470,000 points randomly, $>$540 distinct regions

Can we enumerate employing the previous approach?

Consider the 4×4 matrix

$$M = [m_{ij}]$$

where $m_{ij} \in \{0, 1, a_i\}$ and $m_{4j} = 1$.

- 334 determinants with solutions in the feasible range, $\mathbb{R}^3_{\geq 2}$
- Choose $\sim$470, 000 points randomly, $>$540 distinct regions
- Integral solutions return 188 distinct regions

The Hirsch conjecture

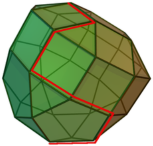
- Stated by Warren M. Hirsch in 1957

The Hirsch conjecture

- Stated by Warren M. Hirsch in 1957
- Puts a bound on the diameter of a polytope with respect to the number of facets

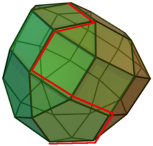
The Hirsch conjecture

- Stated by Warren M. Hirsch in 1957
- Puts a bound on the diameter of a polytope with respect to the number of facets
- Important with regards to the simplex method



The Hirsch conjecture

- Stated by Warren M. Hirsch in 1957
- Puts a bound on the diameter of a polytope with respect to the number of facets
- Important with regards to the simplex method



- Disproven in general by Francisco Santos

Statement of the Hirsch Conjecture

Let $n > d \geq 2$. Let P be a d-dimensional polytope with n facets. Then $\text{diam}(P) \leq n - d$.

Theorem (Naddef, 1989)

$(0, 1)$-*polytopes satisfy the Hirsch conjecture.*

Proof.

Step 1: The diameter is at most d.

- Let u and v be distinct vertices of P.

Proof.

Step 1: The diameter is at most d.

- Let u and v be distinct vertices of P.
- We induct on the dimension d to show the diameter is at most d.

Proof.

Step 1: The diameter is at most d.

- Let u and v be distinct vertices of P.
- We induct on the dimension d to show the diameter is at most d.
- Exhaustively check the case $d = 2$.

Proof.

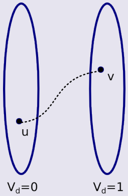
Step 1: The diameter is at most d.

- Let u and v be distinct vertices of P.
- We induct on the dimension d to show the diameter is at most d.
- Exhaustively check the case $d = 2$.

Proof.

Step 1: The diameter is at most d.

- Let u and v be distinct vertices of P.
- We induct on the dimension d to show the diameter is at most d.
- Exhaustively check the case $d = 2$.



Proof ctd.

Step 2: The Hirsch conjecture is satisfied.

- Let u and v be distinct vertices of P.

$\square$

Proof ctd.

Step 2: The Hirsch conjecture is satisfied.

- Let u and v be distinct vertices of P.
- We induct once more on the dimension of P.

□

Proof ctd.

Step 2: The Hirsch conjecture is satisfied.

- Let u and v be distinct vertices of P.
- We induct once more on the dimension of P.
- Both u and v are contained in d facets. We are done if they share a common facet. Else, P contains at least $2d$ facets and we're done.

$\square$

Hirsch conjecture on $(0, 1, a_i)$-polytopes

Theorem

The diameter of a $(0, 1, a_i)$*-polytope is bounded by* $\lfloor \frac{3}{2} d \rfloor$.

This bound is tight!

In dimension 2, consider a hexagon.
In even dimension, consider the product of $\frac{d}{2}$ hexagons.
In odd dimension, consider the prism over the product of $d-1$ hexagons.

A little notation...

For $i \in [d]$, we define

$$V_i^0 := \{v \in \mathrm{vert}(P) : v_i = 0\}$$
$$V_i^1 := \{v \in \mathrm{vert}(P) : v_i = 1\}$$
$$V_i^2 := \{v \in \mathrm{vert}(P) : v_i = a_i\}.$$

Note that V_i^0 and V_i^2 are either empty, or faces of P.

Proof.

We induct on the dimension d.
We want to show for $u, v \in P$, that one of the following two inequalities holds:

$$\delta(u, v) \leq \delta^{d-1} + 1 \tag{1}$$

$$\delta(u, v) \leq \delta^{d-2} + 3 \tag{2}$$

We can assume...

- P is full-dimensional

We can assume...

- P is full-dimensional
- P intersects all facets of the d-orthotope $[0, a_1] \times [0, a_2] \times [0, a_3]$.

Proof ctd.

We will consider the following three cases:

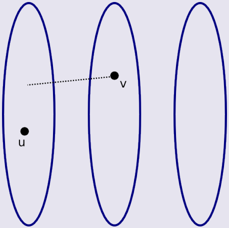
Case 1. There exists an $i \in [d]$ such that $u \in V_0^i$ or V_2^i and $v \in V_1^i$.

Case 2. There exists an $i \in [d]$ such that $u, v \in V_1^i$.

Case 3. For all $i \in [d]$, $|u_i - v_i| = a_i$.

Proof ctd.

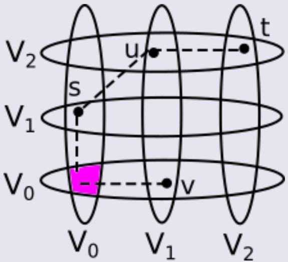
Case 1: There exists an $i \in [d]$ such that $u \in V_0^i$ or V_2^i and $v \in V_1^i$.



Thus $\delta(u,v) \leq \delta^{d-1} + 1$ is satisfied.

Proof ctd.

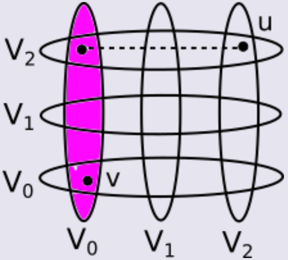
Case 2: There exists an $i \in [d]$ such that $u, v \in V_1^i$.



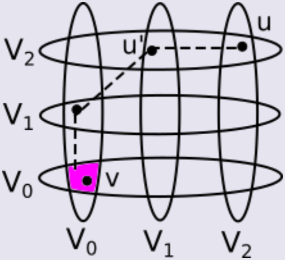
Thus $\delta(u, v) \leq \delta^{d-2} + 3$ is satisfied.

Proof ctd.

Case 3: For all $i \in [d]$, $|u_i - v_i| = a_i$.



Case 3a



Case 3b

Thus, in Case 3a, $\delta(u, v) \leq \delta^{d-1} + 1$ is satisfied.
In Case 3b, $\delta(u, v) \leq \delta^{d-2} + 3$ is satisfied.

References

- Alberto Del Pia and Carla Michini. On the diameter of lattice polytopes. *Discrete and Computational Geometry*, 55(3):681-687, April 2016.
- D. Naddef. The Hirsch conjecture is true for $(0, 1)$-polytopes. *Math. Program.*, 45:109-110, 1989.
- D. Naddef and W.R Pulleyblank. Hamiltonicity in (0,1)-polyhedra. *Journal of Combinatorial Theory, Series B*, 37(1):41 - 52, 1984.