Simple polytopes without small separators

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Separators

- Let $G = (V, E)$ be a connected simple graph (no loops or parallel edges)
- *Definiton:* A separator is a subset C of the vertices of the graph, whose removal partitions the graph to two large disjoint subsets A and B.
- $cn \leq |A| \leq |B| \leq (1 - c)n$ where n is the number of vertices and $0 < c < \frac{1}{2}$ is called *separation constant*

Example

Background

- Lipton & Tarjan (1979) : Planar separator theorem
  - Planar graphs have separators of size $O(\sqrt{n})$ for $c = \frac{1}{3}$
- Miller & al. (1997) : Intersection graphs of balls in d dimensions
  - A k-ply systems have separators of size $O(k^{1/d} n^{1-1/d})$
- Conjecture by Kalai 1991/2004 : Simple d-polytopes have separators of size $O(n^{(d-2)/(d-1)})$

The theorem

- There exist simple d-polytopes whose separators are $\Omega\left(\frac{n}{\log^{3/2} n}\right)$ for all $d \geq 4$ and any c.
- These polytopes can be separate by removing $O\left(\frac{n}{\log n}\right)$ vertices.

Starting point

- Neighborly cubical polytopes (Joswig & Ziegler 2000)
  - $NC_d(m)$ is a d-polytope with k-skeleton of an m-cube, $2k + 2 \leq d$
  - Cubical polytopes, i.e. facets are $(d-1)$-cubes
- Note: They are not neighborly in the usual sense

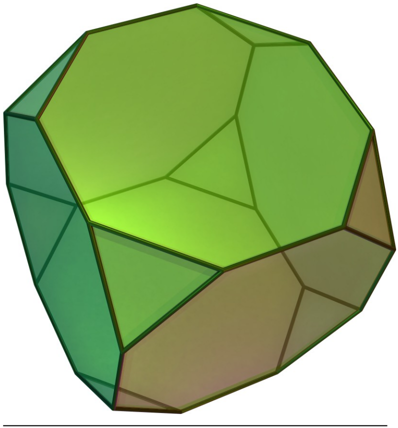
Neighborly cubical polytopes

- Neighborly cubical polytopes arise by perturbing a m-dimensional cube and projecting it down to the desired dimension d.
- The perturbation is done by a neighborly simplicial $d-2$-polytope on $m-1$ vertices.
- There are a lot of such polytopes ($\geq n^{nd/2}$), so neighborly cubical polytopes are not unique at all.

Double truncation, part 1

- Let's focus on $d = 4$, the first interesting case.
- The f-vector is $2^{m-2}(4, 2m, 3m-6, m-2)$
- By truncating all vertices we get the polytope $\mathrm{NC}_4(m)'$ whose facets are
  - simplicial polytopes exposed by the cut (vertex figures of $\mathrm{NC}_4(m)$)
  - vertex-truncated cubes
- This polytope has f-vector $(4m, 14m-24, 11m-22, m+2) \cdot 2^{m-2}$

Truncated cube



Double truncation, part 2

- Now let us truncate the edges of the original polytope
- We end up with the polytope $\mathrm{NC}_4(m)''$ whose facets are
  - Cubes whose vertices and edges have been truncated
  - Simple polytopes, which are the truncated vertex figures
  - Prisms over polygons
- The f-vector of this polytope is $(24m - 48, 48m - 96, 27m - 46, 3m - 2) \cdot 2^{m-2}$
- From the f-vector we see that this polytope is simple

Resulting graph

- $G(NC_4(m)'')$ is the graph of an m-cube Q_m where vertices have been replaced by 3-regular graphs on $6m - 12$ vertices.
  - The 3-regular graphs, which we call *clusters*, are connected with each other through 3 to $m - 1$ edges

Separator

- Let us map an arbitrary separator (A, B, C) of $\mathrm{G}(\mathrm{NC}_4(m)'')$ to Q_m
  - Each cluster is mapped to a single vertex of Q_m
- If all of the vertices in a cluster are in A, insert the vertex of cube into $A' \subset Q_m$
- Do the same for B, and add the rest of the vertices to C'
  - Either (A', B', C') is a separator in Q_m for a suitable c'
  - Or C' is linear size (and we are done)

Separator (cont.)

- Minimal separators in m-cube are of the form $\sum_{i=1}^{m} v_i = k$, where $\boldsymbol{v}$ is the vertex label in a $\{0,1\}^m$ cube.
- For large m, most of the vertices are in layers $m/2 - \lceil \sqrt{m} \rceil, \ldots m/2 + \lceil \sqrt{m} \rceil$
- Approximately $\binom{m}{\frac{m}{2}} = \frac{2^m}{\sqrt{m}}$ vertices have to be removed to separate the cube.
- The separator of $G(NC_4(m)''$ is at least as big as the separator of Q_m.
- Since there are $n := (6m - 12)2^m$ vertices in total, the separator is $\Omega\left(\frac{2^m}{\sqrt{m}}\right) = \Omega\left(\frac{n}{\log^{3/2} n}\right)$

Separator (cont.)

- On the other hand, the clusters are connected on average by less than 6 edges.
- Cutting the m-cube to two $(m-1)$-cubes across a random "axis" therefore separates the graph by removing $6 \cdot 2^m$ vertices.
- This translates to $O(2^m) = O(\frac{n}{\log n})$

Higher dimensions

- Higher dimensional polytopes are created by taking the product with a $(d-4)$-cube for $d > 4$.

Expander graphs (motivation)

- Applications require networks that are robust, i.e. they will tolerate failure of some nodes and still be able to function.
  - For example, computer networks should function even if several parts fail.
  - The failures might be the result of an attack, so we should look at worst case scenarios.
- There is usually a cost for building a connection, so number of edges per vertex should not be large.
- Expander graphs are also useful for randomized algorithms.

Note

- There are different definitions for vertex and edge expansion, but in the case of regular graphs they are qualitatively same

Expander Graphs

- Let $G_i = (V_i, E_i)$ be a sequence of graphs with $|V_i| \to \infty$
- The neighborhood of a set S is $\delta S = \{y \in V(G) | (x, y) \in E(G), x \in S, y \notin S\}$
- G_i is a (vertex) expander if for any $S_i \subset V_i, 2|S_I| \leq |V_i|$

$$h_i = \left\{ \frac{\delta S_i}{S_i} > \epsilon, \epsilon > 0 \right\} \tag{1}$$

- In other words, the separator is size $O(n)$.
- Dividing the graph into two is hard, since any cut will contain vertices proportional to the size of the smaller piece.

Expander Graphs

- Alternative characterization of expander graphs can be achieved algebraically if the graph is d-regular.
- Let $A(G)$ be the normalized adjacency matrix of the graph ($a_{ij} = 1/d$ if edge e_{ij} is present in the graph, 0 otherwise.)
- The largest eigenvalue is 1, since the vector $(1, 1, \ldots, 1)$ is an eigenvector.
- If $|\lambda_2| < 1 - \epsilon$ holds for the graphs, they are expanders.

Expander Graphs

- The algebraic and combinatorial expansion are related by Cheeger inequalities:

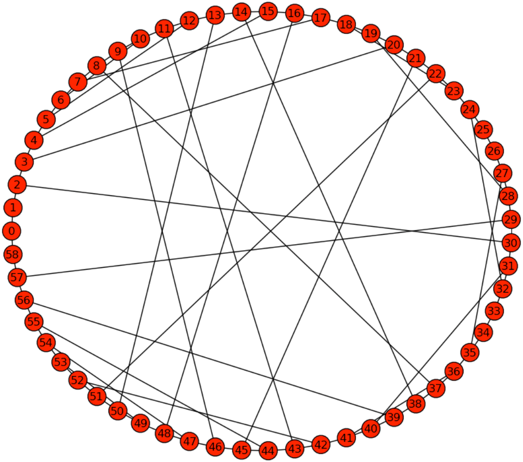
$$\frac{1}{2}(d - d\lambda_2) \leq h(G) \leq \sqrt{2d(d - d\lambda_2)} \tag{2}$$

Expander Graphs

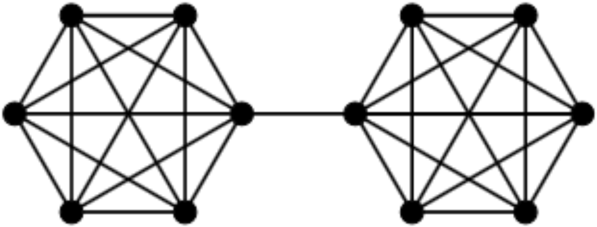
- It seems that the requirements set for the graphs are quite strict and expander graphs would be rare.
- However, asymptotically almost every regular graph is an expander.

Example

- The graph is elements of $\mathbb{Z}_p$, element n is connected to $n+1, n-1$ and n^{-1}.



Non-example



Construction

- Infinite families of expanders can be constructed using so called zig-zag product for graphs.
- Input:
  - A D-regular graph G on vertices $[N]$ with 2nd largest eigenvalue λ_1.
  - d-regular graph H on vertices $[D]$ vertices with 2nd largest eigenvalue λ_2
- Output: d^2 regular graph on DN vertices with 2nd largest eigenvalue at most $\lambda_1 + \lambda_2 + \lambda_2^2$
- Note: The graph might not be simple (it can have self-loops)

Zig-Zag product

- How does it work?
- The edges incident to a given vertex are enumerated from 1 to D (graph G) and 1 to d (graph H).
- For every pair $(i, j) \in \{1, \ldots, d\}^2$ and every pair $(V_1 \in V(G), V_2 \in V(H))$ do the following:
  - Take the path i from V_2 to V_2'.
  - Take the path V_2' from V_1 to V_1'.
  - Take the path j from V_2' to V_2''.
  - Add an edge between (V_1, V_2) and (V_1', V_2'').

Zig-Zag product

- This works since either the steps in the small or the large graph expand well
- This does not necessarily create polytopal graphs, but we hope to be able to modify it

The end

- That's it!