Polyhedral Constructions using the Laplacian Matrix

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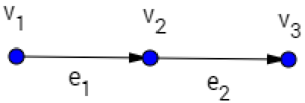
Outline

1. Proof Sketch of the Matrix Tree Theorem
   - Definitions and Notation
2. Dall and Pfeifle's Polyhedral Construction
   - A Polyhedral Proof of the Matrix Tree Theorem (2014)
3. Current Polyhedral Construction
   - Applying Ehrhart Theory

Definitions

- $G := ([n], E)$ is a connected graph with n vertices and $|E| = d$.



- $N :=$ **signed vertex-edge incidence matrix** is the $(n \times d)$-matrix of rank $n-1$.

- $N = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

- $L :=$ **Laplacian** matrix is the $n \times n$ matrix $L = N \cdot N^T$ with eigenvalues $\{0, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}\}$, $\lambda_i \in \mathbb{R}_{>0}$.

- $L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Definition

The zonotope generated by an $(n \times d)$ - matrix M of rank r is the Minkowski sum of the line segments $\mathrm{conv}\{0, M_i\}$, where M_i is the ith column of M.

$$Z(M) = \{\sum_{i=1}^{d} \alpha_i M_i | \alpha_i \in [0,1]\}$$

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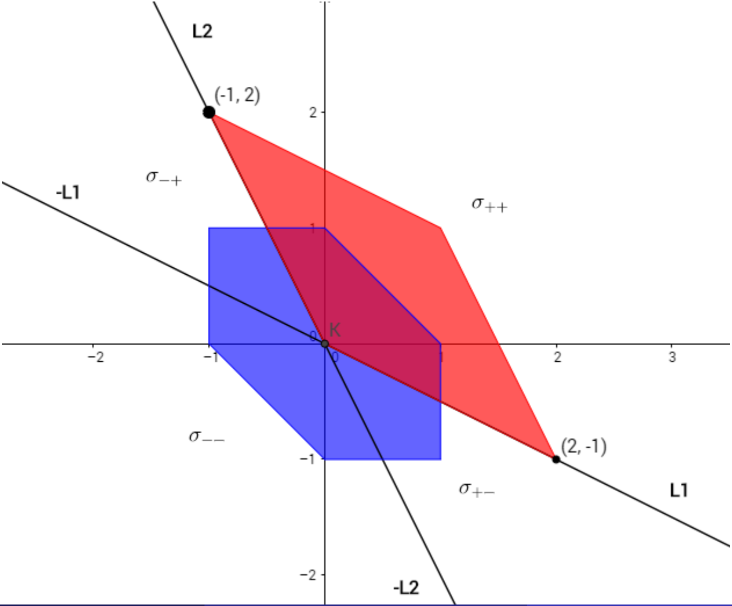
Theorem (Stanley)

$Z(M)$ is the (almost) disjoint union of parallelepipeds indexed by linearly independent columns of M.

Proof Sketch



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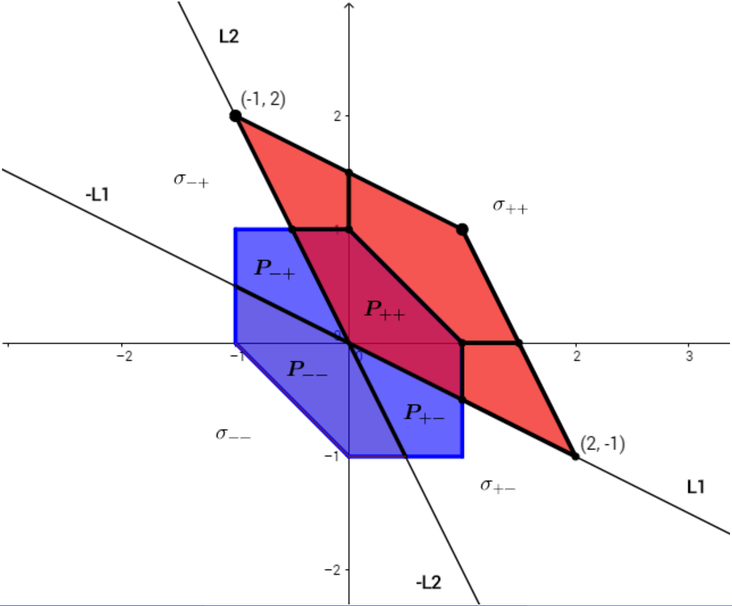

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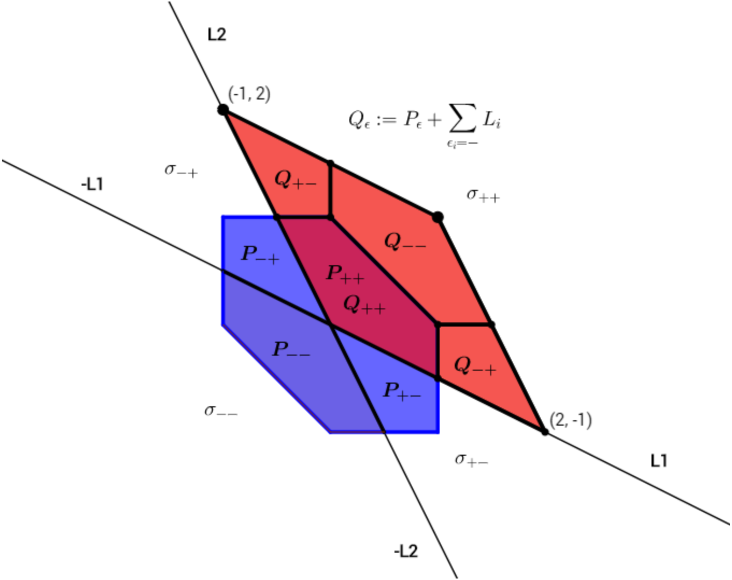

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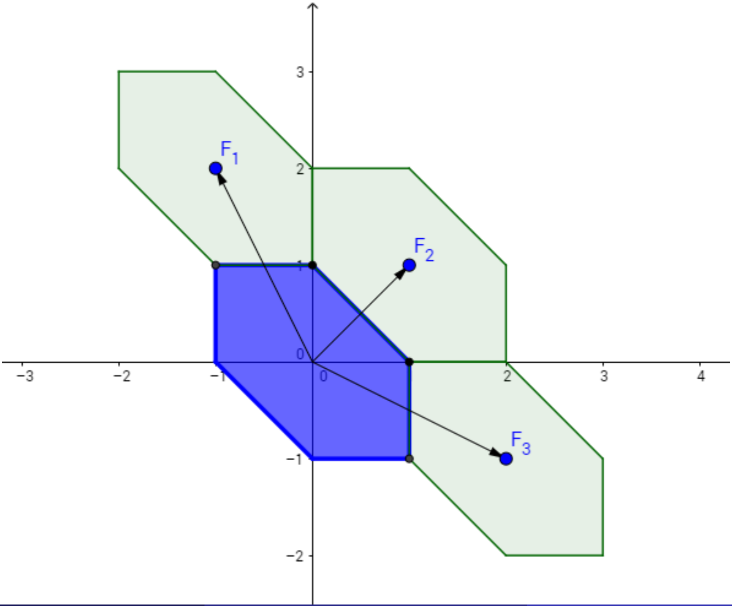

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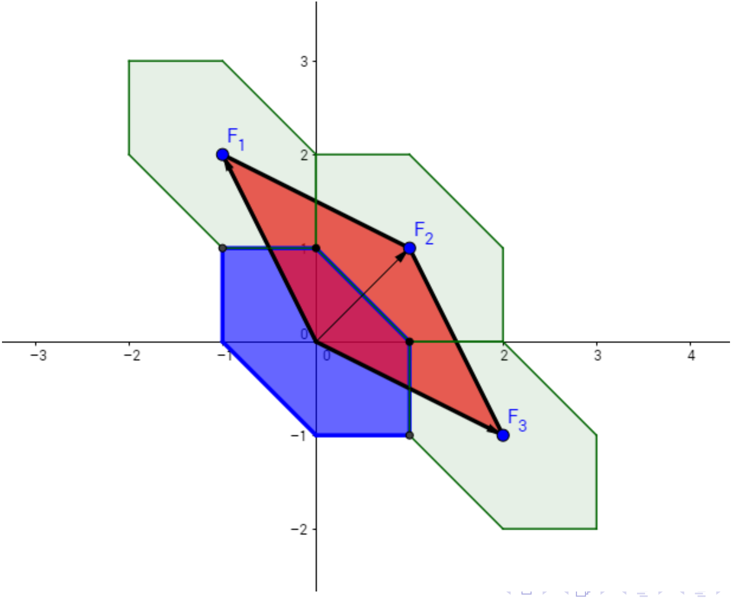

Thanks for listening!