Sublattice index of lattice 3-polytopes

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Definitions

- **Lattice polytope** $P :=$

  convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice).

- **Size** of $P :=$

  number of lattice points in P: $|P \cap \mathbb{Z}^d|$

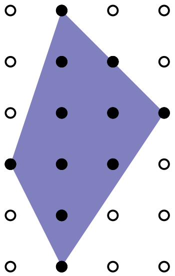
- **Volume** of $P :=$

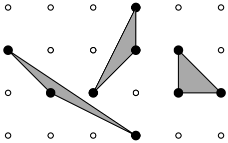
  volume "normalized to the lattice" $=$

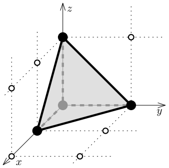
  $d! \times$ Euclidean volume.

  $d+1$ points form a simplex of volume 1 (or **unimodular** simplex) if and only if they affinely span the lattice.

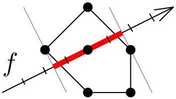
  The volume of a simplex is sometimes called the **determinant**.



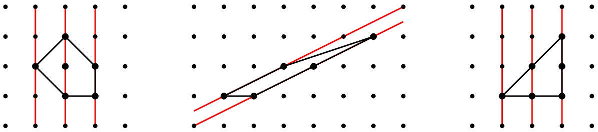




- **Width of P with respect to f,** for a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ = length of the interval $f(P)$



- **Width** of P:= Minimum width of P with respect to a linear NON-CONSTANT, INTEGER functional = minimum lattice distance between two parallel lattice hyperplanes enclosing P



These three parameters of lattice polytopes are strongly connected with their *sublattice index*, as we will later see.

As a previous result, it is useful to know that: For each $n \geq 4$, there are **finitely** many lattice 3-polytopes of size n and width > 1.

(B-Santos, 2014)

Unimodular equivalence

A **unimodular transformation** is a linear integer map $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice. That is,

$$t(x) = A \cdot x + b, \; x \in \mathbb{R}^d$$

for $A \in \mathbb{Z}^{d \times d}$, $\det(A) = \pm 1$ and $b \in \mathbb{Z}^d$.
($t \in GL(n, \mathbb{Z})+$ translations).
Two lattice polytopes P and Q are said **equivalent** or **unimodularly equivalent** if there is an affine unimodular transformation t such that $t(P) = Q$.

Remark

Size, volume, width are invariant modulo unimodular equivalence.

Sublattice index of lattice polytopes

Let $P \subset \mathbb{R}^d$ be a lattice d-polytope and and let $\langle P \cap \mathbb{Z}^d \rangle_{\mathbb{Z}}$ be the affine sublattice generated by $P \cap \mathbb{Z}^d$.

Definition

We call the **sublattice index** of P the index of $\langle P \cap \mathbb{Z}^d \rangle_{\mathbb{Z}}$ as a sublattice of $\mathbb{Z}^d$.
We say that P is **primitive** if its lattice index is 1. That is, if $P \cap \mathbb{Z}^d$ affinely spans $\mathbb{Z}^d$.

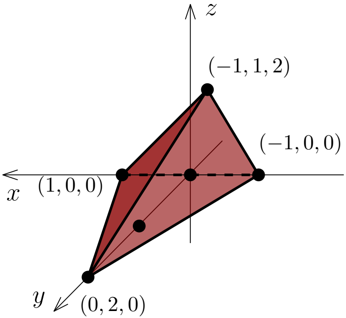
Remarks:

- **Sufficient condition for primitiveness:** If a lattice polytope contains a unimodular d-simplex, then it is primitive. (This is not a *necessary* condition in $d > 2$.)
- In the paper *Ehrhart Theory of Spanning Polytopes* by J. Hofscheier, L. Katthn and B. Nill, they call spanning what we call primitive.
- **In this talk, "index" will stand for "sublattice index"**.

In dimension 1 and 2, every lattice polytope contains a unimodular simplex, and hence every 1 or 2-dimensional lattice polytope is primitive.

In dimension 3, the following polytope has index 2:

$$P \cap \mathbb{Z}^3 = \{(1,0,0),(0,0,0),\\ (-1,0,0),(0,1,0),\\ (0,2,0),(-1,1,2)\}$$



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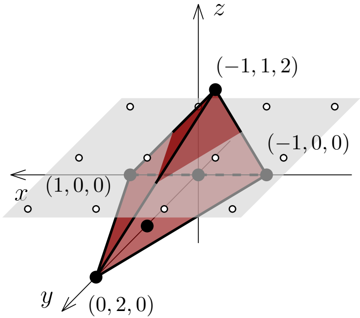

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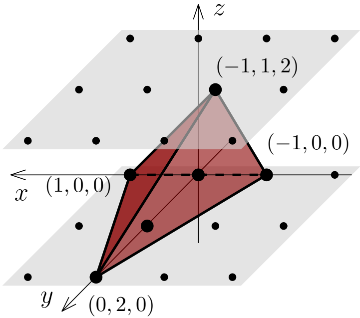


$$\langle P \cap \mathbb{Z}^3 \rangle_{\mathbb{Z}} = \left\{ (x,y,z) \in \mathbb{Z}^3 \mid z \equiv 0 \mod 2 \right\} \quad \Longrightarrow \quad \textbf{Index } 2$$

So, *how does the index of lattice 3-polytopes behave?*

GOAL: characterize the index of lattice 3-polytopes

STARTING POINT: We have a full classification of lattice 3-polytopes of width > 1 and sizes up to 11 **(B-Santos, 2016)**.

We computed their indices:

size	**5**	**6**	**7**	**8**	**9**	**10**	**11**
index 1, no unimodular tetrahedron	2	-	-	-	-	-	-
index 2	-	2	8	14	15	19	24
index 3	1	3	2	3	3	4	4
index 5	1	-	-	-	-	-	-

The table SUGGESTS:

This is what actually happens in larger sizes!!

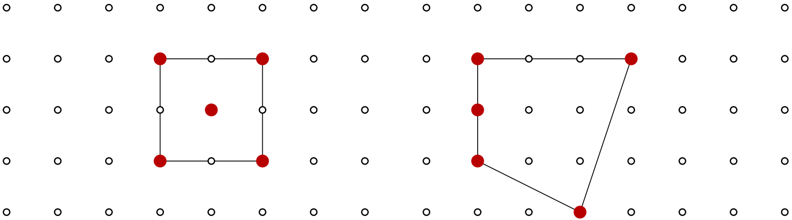
- Only **indices** 1, 2, 3 **or** 5 appear.
- There is a unique polytope of **index** 5.
- The number of those of **index** 3 grows linearly with size.
- The number of those of **index** 2 grows quadratically with size.
- Except for two polytopes of size 5, all the **primitive** ones contain a **unimodular tetrahedron**.

SUBLATTICE INDEX OF LATTICE POINT CONFIGURATIONS

Let $A \subset \mathbb{Z}^d$ be a point configuration and let $\langle A \rangle_{\mathbb{Z}}$ be the affine sublattice generated by A.

Definition

We call the **sublattice index** of A the index of $\langle A \rangle_{\mathbb{Z}}$ as a sublattice of $\mathbb{Z}^d$. We say that A is **primitive** if its lattice index is 1. That is, if A contains an affine basis of $\mathbb{Z}^d$.



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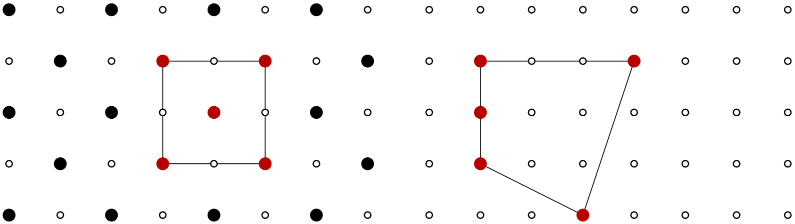

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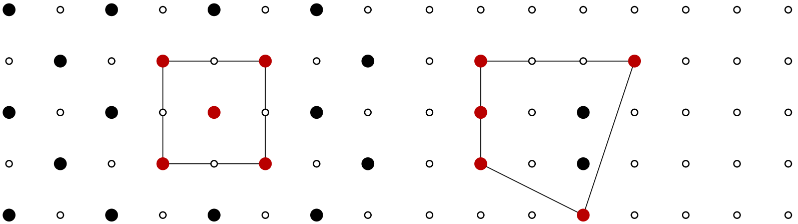

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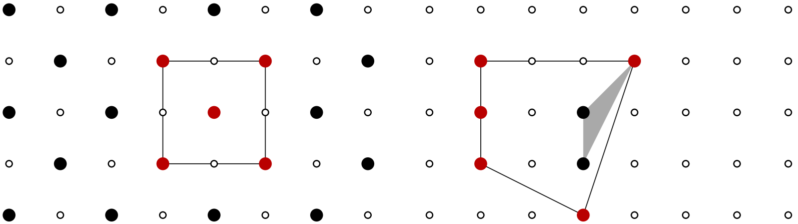

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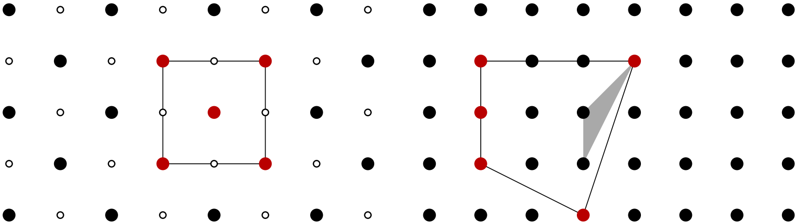


Index = gcd of volumes

The **volume vector** of A is an integer vector that records the determinants of all $(d+1)$-tuples of points in A. We denote it by $v(A)$.

Lemma

The sublattice index of A equals the greatest common divisor of the entries in $v(A)$.

Proof.

• Index divides $gcd(v(A))$

index $|$ determinant of points in $\langle A \rangle_{\mathbb{Z}} \implies$
$\implies$ index $|$ determinant of points in A=entry of $v(A)$

• $gcd(v(A))$ divides index

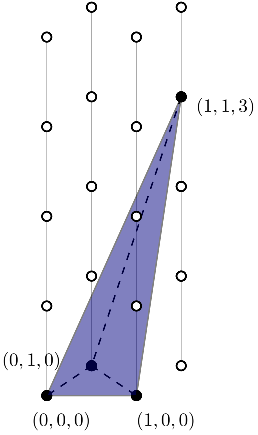
Let $\Delta = \text{conv}\{p_1, \ldots, p_{d+1}\}$ with $p_i \in \langle A \rangle_{\mathbb{Z}}$ and $\det(\Delta) = \text{index}$.
$p_i = \text{integer combination}(A) \implies$
$\implies index = \det(\Delta) = \text{int comb(determinant of points in } A) =$
$= \text{int comb(coordinates of } v(A))$, which is a multiple of $\gcd(v(A))$ $\square$

Index = gcd of volumes

Example

Let Δ be an *empty* d-simplex ("empty means $\mathrm{vert}(\Delta) = \Delta \cap \mathbb{Z}^d$") of volume q.
Then its sublattice index is q.

- We compute indices by computing the volume vectors.
- To conclude that A is primitive, it suffices to check that there are <u>some</u> simplices in $\binom{A}{d+1}$ whose gcd of volumes is 1.
- To conclude that $\mathrm{index}(A) = q > 1$, it suffices to check that <u>all</u> simplices in $\binom{A}{d+1}$ have volume 0 or a multiple of q, and that there are <u>some</u> of them whose gcd of volumes is q.



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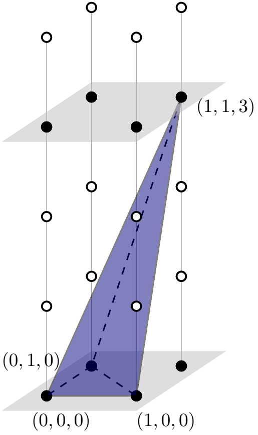


Index of A divides index of $B \subseteq A$

Lemma

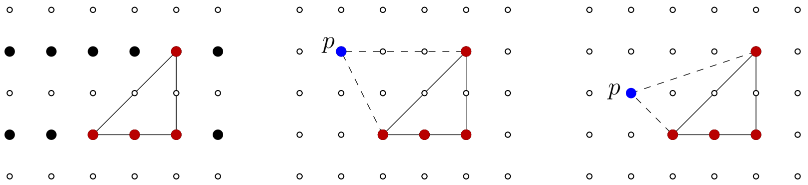
The index of A divides the index of any subconfiguration $B \subseteq A$.

Proof.

Enough to prove it for $B = A \setminus \{p\}$.

- If $p \in \langle B \rangle_{\mathbb{Z}}$, then $\langle A \rangle_{\mathbb{Z}} = \langle B \rangle_{\mathbb{Z}}$ and the index does not change.
- If $p \notin \langle B \rangle_{\mathbb{Z}}$, then $\langle B \rangle_{\mathbb{Z}}$ is a subgroup of $\langle A \rangle_{\mathbb{Z}}$ and hence the index of B multiplies the index of A.

$\square$



That is, *the bigger the size, the smaller the index.*

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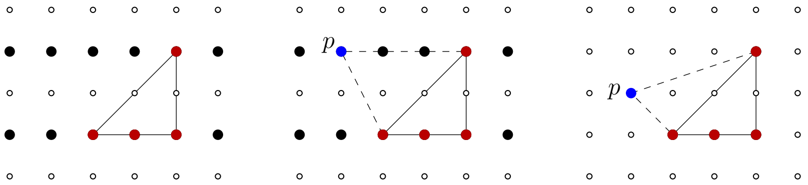

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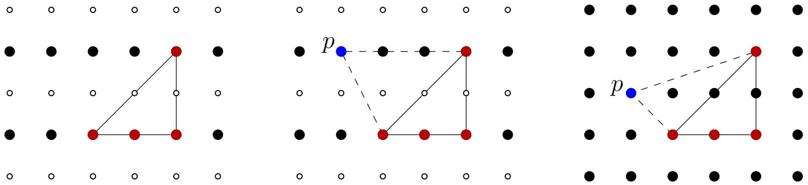


That is, <u>*the bigger the size, the smaller the index.*</u>

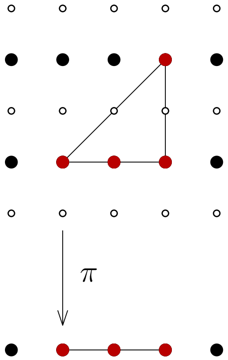
Index of A is multiple of index of projection $\pi(A)$

Lemma

Let $A \subset \mathbb{Z}^d$ be a lattice point set and let $\pi : \mathbb{Z}^d \to \mathbb{Z}^s$, for $s < d$, be a lattice projection. Then the sublattice index of $\pi(A)$ divides the sublattice index of A.

Proof.

A lattice projection is a surjective group homomorphism. Then, $\langle \pi(A) \rangle_{\mathbb{Z}} = \pi(\langle A \rangle_{\mathbb{Z}})$ and the index of $\langle \pi(A) \rangle_{\mathbb{Z}}$ as a subgroup of $\pi(\mathbb{Z}^d) = \mathbb{Z}^s$ divides the index of $\langle A \rangle_{\mathbb{Z}}$ as a subgroup of $\mathbb{Z}^d$. $\square$



That is, *projecting to smaller dimension decreases the index.*

SUBLATTICE INDEX OF LATTICE 3-POLYTOPES

We want to study the sublattice index of lattice 3-polytopes. We separate three cases and deal with them separately:

(I) **Polytopes of width one**

They are easy to understand.

(II) **Polytopes of width > 1 and size ≤ 11**

We take the information from our classification.

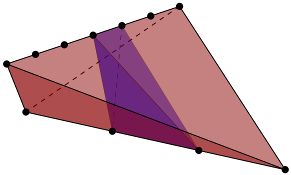
(III) **Polytopes of width > 1 and size > 11**

We use parts (I) and (II), and properties of the sublattice index under projection or inclusion.

(I) Index of polytopes of width 1

Lattice 3-polytopes of width one have all their lattice points distributed in two consecutive parallel lattice hyperplanes

- If the polytope has three non-collinear point in one of these planes, then in particular there is a unimodular triangle with points of P in this plane, which forms a unimodular tetrahedron with any of the points in the other plane.
- If not, then P consists of lattice points along two lattice segments and its index is that of any emtpy tetrahedron contained in it.



Corollary

Let P be a lattice 3-polytope of width one. Then either P contains a unimodular tetrahedron or P is equivalent to

$$\mathrm{conv}\{(0,0,0),(k,0,0),(0,0,1),(rp,rq,1)\}$$

for some $k, r, > 0$, $\gcd(p,q)=1$, of index $q>1$.

(II) Polytopes of width > 1 & size ≤ 11:

INDEX 1 (primitive polytopes)

Theorem

Among all lattice 3-polytopes of width > 1 and sizes 5 to 11, the following two are the only primitive polytopes that DO NOT contain a unimodular tetrahedron: they are both terminal tetrahedra (of size 5):

- $\mathrm{conv}\{(0,0,0),(1,0,0),(0,0,1),(2,7,1),(-1,-2,-1)\}$, *with four empty tetrahedra of volumes* 2, 3, 5 *and* 7.
- $\mathrm{conv}\{(0,0,0),(1,0,0),(0,0,1),(3,7,1),(-2,-3,-1)\}$, *with four empty tetrahedra of volumes* 3, 4, 5 *and* 7.

(II) Polytopes of width > 1 & size ≤ 11:

INDEX 5

Theorem

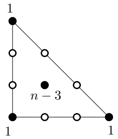
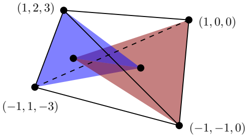
Among all lattice 3-polytopes of width > 1 and sizes 5 to 11, the following is the only polytope of index 5:

$$\operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(2,5,1),(-3,-5,-2)\},$$

a terminal tetrahedron (size 5), with four empty tetrahedra of volume 5.

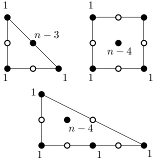
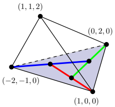
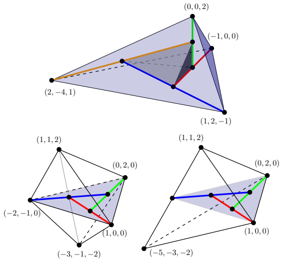
(II) Polytopes of width > 1 & size ≤ 11:

INDEX 3

size	**5**	**6**	**7**	**8**	**9**	**10**	**11**
projects to 	1	2	2	3	3	4	4
exceptions	-		-	-	-	-	-

(II) Polytopes of width > 1 & size ≤ 11:

INDEX 2

size	5	6	7	8	9	10	11
projects to 	-	2	7	11	15	19	24
exceptions	-	-			-	-	-

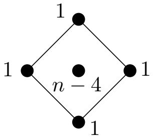
(III) For size > 11 and width > 1, index is $1, 2, 3$

Lemma (B-Santos, 2016)

Let P be a lattice 3-polytope of size $n > 11$ and width > 1. Then

1. *There exists a vertex $u \in \mathrm{vert}(P)$ such that $P^u := \mathrm{conv}(P \cap \mathbb{Z}^3 \setminus \{u\})$ has width > 1,*
2. *or P admits a lattice projection to the following 2-dimensional configuration.*

   (The labels in the points indicate how many (consecutive) lattice points of P project to it)



Theorem

Every lattice 3-polytope of width > 1 and size $n > 11$ has index 1, 2 or 3.

Proof.

True for $n = 11$ (PART (II)). Let P be of size $n > 11$ and width > 1.

- If P is as in part (1), then by induction the index of P^u is 1, 2 or 3. Since P^u is a subconfiguration of P, so is the index of P.
- If P is as in part (2), it contains a unimodular tetrahedron (index 1).

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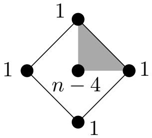

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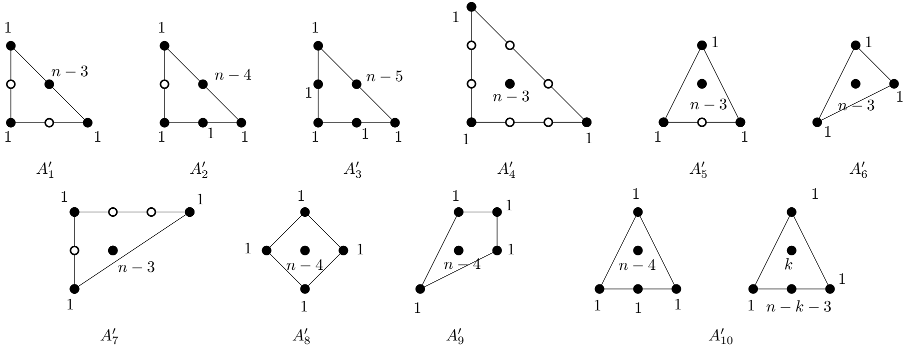
(III) Characterizing polytopes with each index

Lemma (B-Santos, 2016)

Let P be a lattice 3-polytope of size $n > 11$ and width > 1. Then:

1. *There exist vertices $u, v \in \mathrm{vert}(P)$ such that P^u and P^v have width one and $P^{uv} := \mathrm{conv}(P \cap \mathbb{Z}^3 \setminus \{u, v\})$ is 3-dimensional.*
2. *or P admits a lattice projection to one of the following 2-dimensional configurations. (The labels in the points indicate how many (consecutive) lattice points of P project to it)* (**SPIKED**)

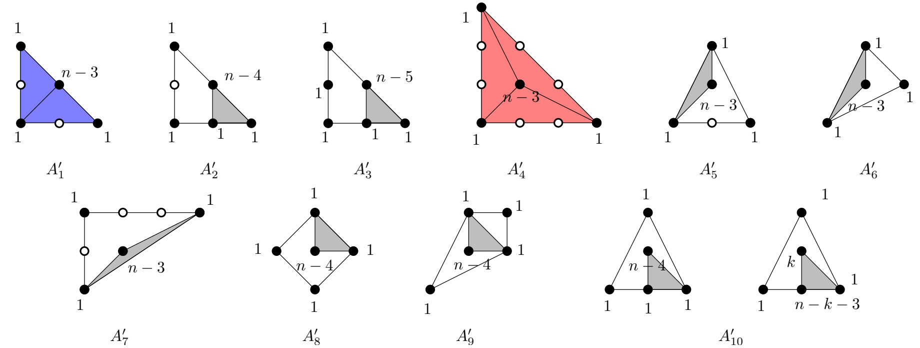


(III) Index of spiked polytopes

Lemma

Let P be a spiked 3-polytope:

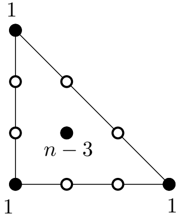
1. *If P is spiked projecting to A'_1, its index is 2.*
2. *If P is spiked projecting to A'_4, its index is 3.*
3. *If P is spiked projecting to any other A'_i, then P contains a unimodular tetrahedron.*



(III) Characterizing polytopes of index 3

Theorem

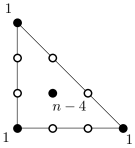
Let P be a lattice 3-polytope of width > 1, size $n > 11$ and index 3, then P admits a lattice projection to the following 2-dimensional configuration:



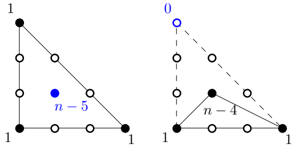
Proof.

True for $n = 11$ and for P of size > 11 and spiked.
Otherwise, P has two vertices u, v such that P^u and P^v have width > 1 and P^{uv} is 3-dimensional. Then by induction, P^u and P^v project to the configuration:

P^u and P^v:



Options for P^{uv}:



(III) Characterizing polytopes of index 3

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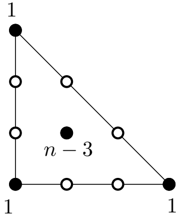

Proof.

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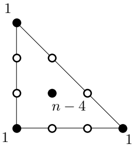
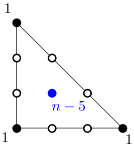
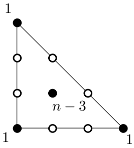
P^u and P^v: 

If P^{uv}: 

$\Longrightarrow$ P: 

$\square$

(III) Characterizing polytopes of index 3

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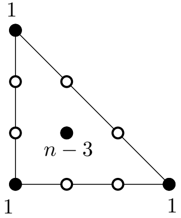
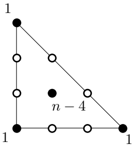


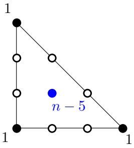
Proof.

True for $n = 11$ and for P of size > 11 and spiked.
Otherwise, P has two vertices u, v such that P^u and P^v have width > 1 and P^{uv} is 3-dimensional. Then by induction, P^u and P^v project to the configuration:

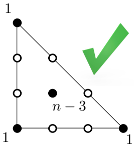
P^u and P^v:



If P^{uv}:



$\Longrightarrow$ P:

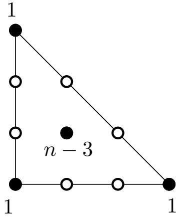


$\square$

(III) Characterizing polytopes of index 3

Theorem

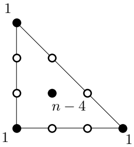
Let P be a lattice 3-polytope of width > 1, size $n > 11$ and index 3, then P admits a lattice projection to the following 2-dimensional configuration:



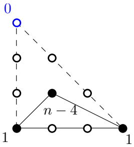
Proof.

True for $n = 11$ and for P of size > 11 and spiked.
Otherwise, P has two vertices u, v such that P^u and P^v have width > 1 and P^{uv} is 3-dimensional. Then by induction, P^u and P^v project to the configuration:

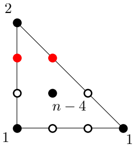
P^u and P^v:



If P^{uv}:



$\Longrightarrow$ P:

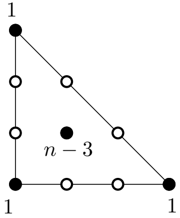


□

(III) Characterizing polytopes of index 3

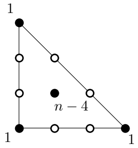
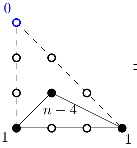
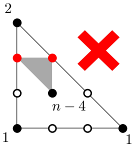
Theorem

Let P be a lattice 3-polytope of width > 1, size $n > 11$ and index 3, then P admits a lattice projection to the following 2-dimensional configuration:



Proof.

True for $n = 11$ and for P of size > 11 and spiked.
Otherwise, P has two vertices u, v such that P^u and P^v have width > 1 and P^{uv} is 3-dimensional. Then by induction, P^u and P^v project to the configuration:

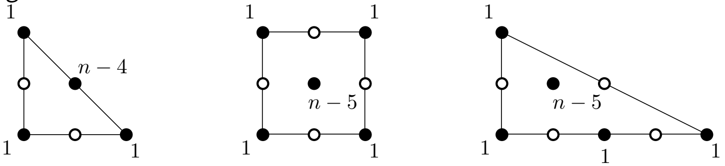
P^u and P^v:  If P^{uv}:  $\Longrightarrow$ P: 

□

(III) Characterizing polytopes of index 2

Theorem

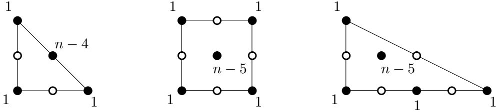
Let P be a lattice 3-polytope of width > 1, size $n > 11$ and index 2, then P admits a lattice projection to one of the following 2-dimensional configurations:



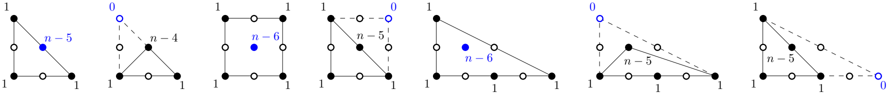
Proof.

Proof is analogous.

P^u and P^v:



$(P^u, P^{u,v})$ and $(P^v, P^{u,v})$:



(III) Characterizing primitive polytopes

Theorem

Let P be a primitive 3-polytope of width > 1 and size $n > 11$. Then P contains a unimodular tetrahedron.

Proof.

True for $\underline{n = 11}$ and for P of size > 11 and spiked.
Otherwise P has vertices u, v so that P^u and P^v are of width > 1 and such that $P^{u,v}$ is 3-dimensional. Let I_u, I_v and I_{uv} be indices of P^u, P^v and $P^{u,v}$.

If $I_u = 1$, by induction P^u contains a unimodular tetrahedron, and so does P. Same if $I_v = 1$.

So assume $I_u, I_v > 1$. Notice that $P^{u,v}$ is a lattice 3-polytope of index $I_{uv} > 1$ a multiple of I_1 and I_2. Let us see that $I_{uv} = I_u = I_v$.
We have two possibilities:

- If $P^{u,v}$ has width > 1, then by induction $I_{uv} = 2, 3 \Longrightarrow I_{uv} = I_u = I_v$
- If $P^{u,v}$ has width one, then (by an extra result) $\Longrightarrow I_{uv} = I_u$ and $I_{uv} = I_v$

$\Longrightarrow$ The quotient sublattices $\mathbb{Z}^3/P^u$, $\mathbb{Z}^3/P^v$ and $\mathbb{Z}^3/P^{u,v}$ (and, therefore, $\mathbb{Z}^3/P$) are the same $\Longrightarrow$ P is not primitive (CONTRADICTION).

$\square$

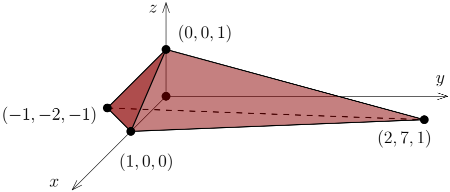
SUMMARY OF RESULTS

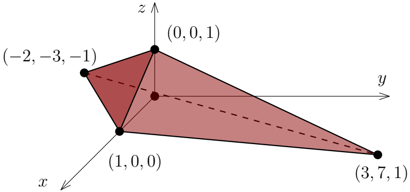
Primitive polytopes

Corollary

The following two are the only primitive 3-polytopes that DO NOT contain a unimodular tetrahedron:

- $\mathrm{conv}\{(0,0,0),(1,0,0),(0,0,1),(2,7,1),(-1,-2,-1)\}$, *a terminal tetrahedron (size* 5*), with four empty tetrahedra of volumes* 2, 3, 5 *and* 7.
- $\mathrm{conv}\{(0,0,0),(1,0,0),(0,0,1),(3,7,1),(-2,-3,-1)\}$ *a terminal tetrahedron (size* 5*), with four empty tetrahedra of volumes* 3, 4, 5 *and* 7.





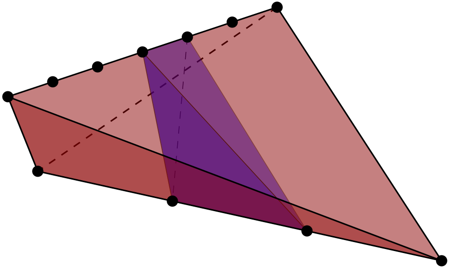
Indices of lattice 3-polytopes

Corollary

Let P be a lattice 3-polytope of $\underline{\text{index } q \notin \{1,2,3,5\}}$. Then P has width one.

Corollary

If P is a lattice polytope of $\underline{\text{width one and index } q > 1}$, then it consists of lattice points along two lattice segments and all its empty tetrahedra have volume q:



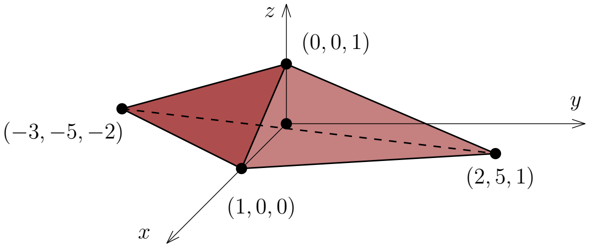
Polytopes of width > 1: **INDEX 5**

Corollary

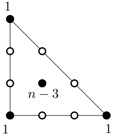
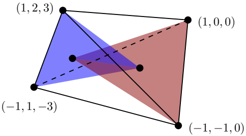
The following is the only lattice 3-polytope of *width > 1 and index 5*:

$$\operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(2,5,1),(-3,-5,-2)\},$$

a terminal tetrahedron (size 5), with four empty tetrahedra of volume 5.



Polytopes of width > 1: **INDEX 3**

<table>
<tr><th>size</th><th>5</th><th>6</th><th>$n \geq 7$</th></tr>
<tr><td>projects to

</td><td>1</td><td>2</td><td>$\left\lfloor \frac{n}{2} \right\rfloor - 1$</td></tr>
<tr><td>exceptions</td><td>-</td><td>
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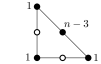
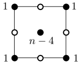
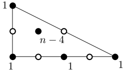
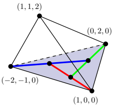
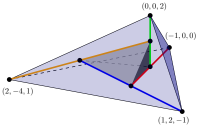
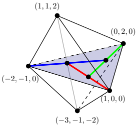
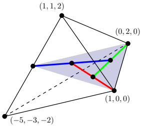
Polytopes of width > 1: **INDEX 2**

size	**6**	**7**	**8**	**n $\geq$ 9**
projects to				
	2	2	3	$\left\lfloor \frac{n}{2} \right\rfloor - 1$
	-	4	7	$\frac{1}{2}\left[\binom{n-4}{2} - \left\lfloor \frac{n-4}{2} \right\rfloor\right] + n - 4$
	-	1	1	$\left\lfloor \frac{n-1}{2} \right\rfloor - 1$
exceptions	-		  	-

Thank you for your attention!!

Mónica Blanco and Francisco Santos, *On the enumeration of lattice 3-polytopes*, Preprint, 31 pages, January 2016. `arXiv:1601.02577`. An extended abstract has appeared as "Enumerating lattice 3-polytopes", Electr. Notes Disc. Math. 54 (October 2016), 133–138. DOI: 10.1016/j.endm.2016.09.024.

Johannes Hofscheier, Lukas Katthän and Benjamin Nill, *Ehrhart Theory of Spanning Polytopes*, Preprint, August 2016. `arXiv:1608.03166`.