The intersection ring of matroids

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Workshop on Convex Polytopes

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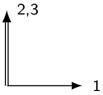
Matroid terminology

- A *matroid* M (think: list of vectors) on a set $E := \{1, \dots, n\}$ is a collection $\mathcal{C}$ of subsets of E, called *circuits* (think: minimal dependent subsets), fulfilling a *circuit exchange axiom*.



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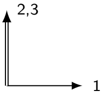

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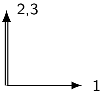

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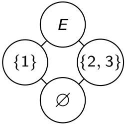

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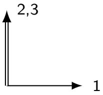

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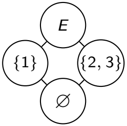

Convention: All matroids are loopfree (think: no zero vectors), i.e. $\varnothing$ is a flat.

A vector space of matroids

- Let $\mathfrak{C}_{r,n}$ be the set of all chains of subsets of E of the form

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$$(w_M)_{\mathcal{L}} := \begin{cases} 1, & \text{if } \mathcal{L} \text{ is a chain of flats of } M, \\ 0, & \text{otherwise.} \end{cases}$$

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- This defines a vector space $\mathbb{M}_{r,n}$, which is the subspace of $\mathbb{R}^{\mathfrak{C}_{r,n}}$ generated by the vectors w_M.

Some geometric intuition

- For a set $F \subseteq E$, let $v_F := \sum_{i \in F} e_i \in \mathbb{R}^n/(1,\ldots,1)$.

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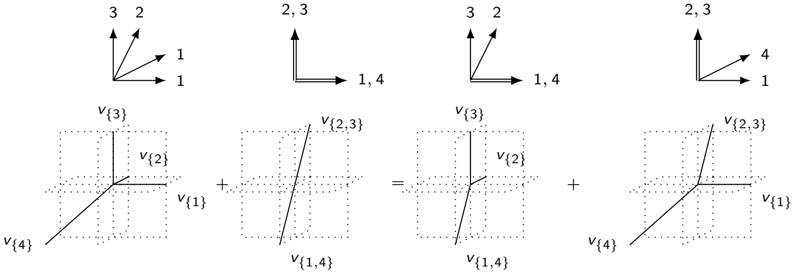

Note: $v_{\{4\}} = (0,0,0,1) \equiv (-1,-1,-1,\not{0})$.

Cyclic flats and nested matroids

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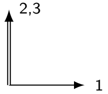

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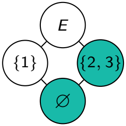

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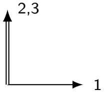

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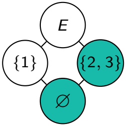

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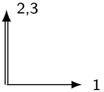

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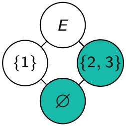

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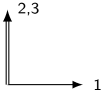

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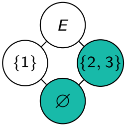

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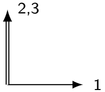

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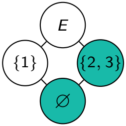

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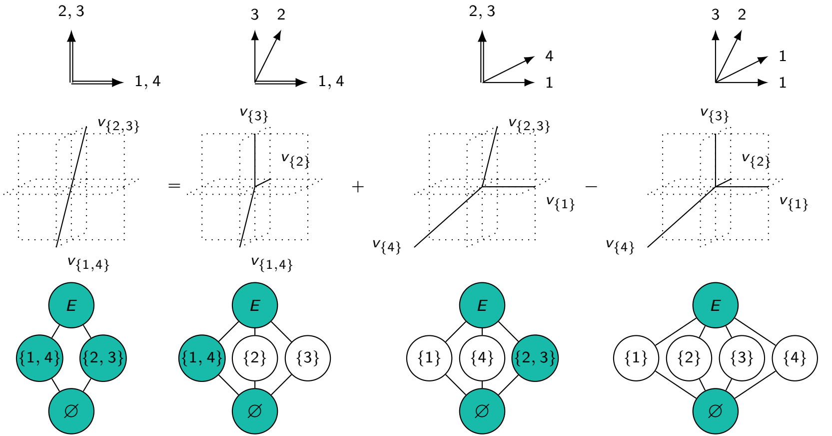
Theorem (H)

The nested matroids form a basis of $\mathbb{M}_{r,n}$.

An example



2, 3
1, 4
3
2
1, 4
2, 3
4
1
3
2
1
1
$v_{\{2,3\}}$
$v_{\{1,4\}}$
=
$v_{\{3\}}$
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$v_{\{1,4\}}$
+
$v_{\{2,3\}}$
$v_{\{1\}}$
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−
$v_{\{3\}}$
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E
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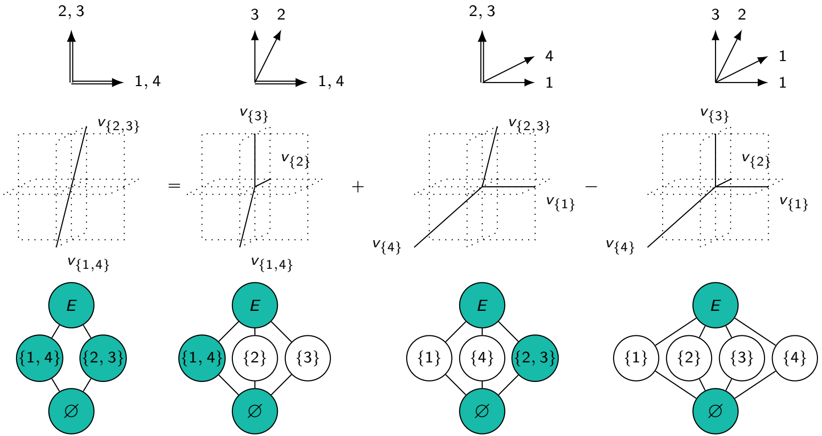

No. of bases:

$$4 = 5 + 5 - 6$$

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The $\mathcal{G}$-invariant induces a linear map

$$\mathbb{M}_{r,n} \rightarrow \mathbb{R}^{\binom{n}{r}} \quad .$$

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  - Addition = "Weighted sums of sets".
  - Multiplication = *Stable intersection* $\cap_{\text{st}}$.

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Theorem (Speyer '08)

The stable intersection of two Bergman fans is

$$B(M) \cap_{\text{st}} B(M) = \begin{cases} B(M \wedge M'), \text{if } M \wedge M' \text{ is loopfree,} \\ 0, \text{otherwise.} \end{cases}$$

The intersection of matroids

- We thus obtain a ring $\mathbb{M}_n := \bigoplus_{r=1}^{n} \mathbb{M}_{r,n}$, where the product is defined by

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Proposition (H)

Every nested matroid is a product of corank one matroids, so $\mathbb{M}_n$ is generated in corank one.

Counting nested matroids

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Corollary

$\mathbb{M}_n$ fulfills Poincaré duality, i.e.
$\mathbb{M}_{r,n} \cong \mathrm{Hom}(\mathbb{M}_{n-r+1,n}, \mathbb{R})$ ($\cong \mathbb{M}_{n-r+1,n}$) via

$$M \mapsto (N \mapsto M \cdot N) \ .$$

McMullen's polytope algebra

McMullen defined a *polytope algebra* Π_n: It is the algebra generated by symbols $[P]$ for each polytope in $\mathbb{R}^n$, modulo translations and the identity

$$[P \cup Q] = [P] + [Q] - [P \cap Q] \ ,$$

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Theorem (H)

$\mathbb{M}_{r,n} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the subalgebra of Π_{n-1} generated by matroid polytopes.

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- Since many invariants are linear maps, extremality questions can be phrased as linear programs.

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- These polytopes get large very quickly.
- (partial) f-vectors of the smallest non-trivial examples:
  - $(n, r) = (4, 2) : (14, 85, 298, 673, 1029, 1085, 785, 378, 113, 18)$.
  - $(n, r) = (5, 2) : (51, 1105, 14075, ? \ldots ?, 319)$.
  - $(n, r) = (5, 3) : (106, 5365, ? \ldots ?, 394)$.

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