Toric Fano varieties associated to building sets

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Algebraic and Geometric Combinatorics on Convex Polytopes

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Table of Contents

1. Toric varieties and fans

Definition

An n-dimensional toric variety is a normal algebraic variety X over $\mathbb{C}$ containing $(\mathbb{C}^*)^n$ as an open dense subset, s.t. the natural action $(\mathbb{C}^*)^n \curvearrowright (\mathbb{C}^*)^n$ extends to an action on X.

Examples

$(\mathbb{C}^*)^n, \mathbb{C}^n, \mathbb{P}^n$ are toric varieties.

Definition

A rational strongly convex polyhedral cone is a cone $\sigma \subset \mathbb{R}^n$ generated by finitely many vectors in $\mathbb{Z}^n$ which does not contain any non-zero linear subspace of $\mathbb{R}^n$.

Definition

A fan in $\mathbb{R}^n$ is a non-empty finite set Δ of such cones satisfying the following conditions:

- If $\sigma \in \Delta$, then each face of σ is in Δ.
- If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of each.

Fact

$$\{\text{fans in } \mathbb{R}^n\} \xleftrightarrow{1:1} \{n\text{-dimensional toric varieties}\},$$
$$\Delta \mapsto X(\Delta).$$

Construction of a toric variety $X(\Delta)$ from a fan Δ.

Step 1 (affine toric varieties)

First we construct an affine toric variety U_σ for each $\sigma \in \Delta$.

- $\sigma^\vee = \{u \in \mathbb{R}^n \mid \langle u, v\rangle \geq 0 \ \forall v \in \sigma\}$: the dual of σ.
- $\sigma^\vee \cap \mathbb{Z}^n$ is a commutative monoid.
- The monoid ring $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ is a finitely generated integral domain. So we put $U_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$.

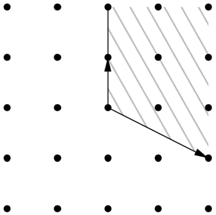
Step 2 (gluing)

Let τ be a face of σ and let $\tau \to \sigma$ be the inclusion.

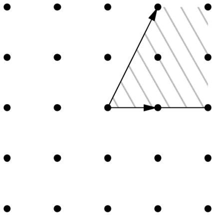
- $\rightsquigarrow$ a monoid homomorphism $\sigma^\vee \cap \mathbb{Z}^n \to \tau^\vee \cap \mathbb{Z}^n$.
- $\rightsquigarrow$ an open immersion $U_\tau \to U_\sigma$.
- Gluing $\{U_\sigma \mid \sigma \in \Delta\}$, we obtain the toric variety $X(\Delta)$.

Example

$\sigma = \mathbb{R}_{\geq 0}(2e_1 - e_2) + \mathbb{R}_{\geq 0}e_2 \subset \mathbb{R}^2$
$\rightsquigarrow \sigma^\vee = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1 + 2e_2)$
$\rightsquigarrow \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{Z}_{\geq 0}e_1 + \mathbb{Z}_{\geq 0}(e_1 + e_2) + \mathbb{Z}_{\geq 0}(e_1 + 2e_2)$.



$\sigma = \mathbb{R}_{\geq 0}(2e_1 - e_2) + \mathbb{R}_{\geq 0}e_2$



$\sigma^\vee = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(e_1 + 2e_2)$

$\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2] = \mathbb{C}[X, XY, XY^2] = \mathbb{C}[U, V, W]/(UW - V^2)$. Therefore $U_\sigma = \mathrm{Spec}\,\mathbb{C}[U, V, W]/(UW - V^2)$.

Definition

- Δ is nonsingular $\Leftrightarrow$ every cone of Δ is generated by a part of a basis for $\mathbb{Z}^n$.
- Δ is complete $\Leftrightarrow \bigcup_{\sigma \in \Delta} \sigma = \mathbb{R}^n$.

Fact

- $X(\Delta)$ is nonsingular $\Leftrightarrow$ Δ is nonsingular.
- $X(\Delta)$ is complete $\Leftrightarrow$ Δ is complete.

2. Toric varieties associated to building sets

S: a nonempty finite set.

Definition

A building set on S is a finite set B of nonempty subsets of S satisfying the following conditions:

- If $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.
- For every $i \in S$, we have $\{i\} \in B$.

$B_{\max}$: the set of all maximal (by inclusion) elements of B.
An element of $B_{\max}$ is called a B-component.
B is connected $\Leftrightarrow B_{\max} = \{S\}$.
$B|_C = \{I \in B \mid I \subset C\}$: the restriction of B to C ($\emptyset \neq C \subset S$).
$B = \bigsqcup_{C \in B_{\max}} B|_C$ for any building set B.

$G = (V(G), E(G))$: a finite simple graph, that is, a finite graph with no loops and no multiple edges.

Definition

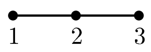
For $I \subset V(G)$, we define the induced subgraph $G|_I$ by

$$V(G|_I) = I, E(G|_I) = \{\{v, w\} \in E(G) \mid v, w \in I\}.$$

$B(G) = \{I \subset V(G) \mid G|_I \text{ is connected}, I \neq \emptyset\}$ is a building set on $V(G)$. We call $B(G)$ the graphical building set.

Example

Let P_3 be the path graph with 3 nodes.



Then $B(P_3) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.

Definition

A nested set of B is a subset $N \subset B \setminus B_{\max}$ satisfying the following conditions:

- If $I, J \in N$, then we have either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$.
- For any $k \geq 2$ and for any pairwise disjoint $I_1, \ldots, I_k \in N$, we have $I_1 \cup \cdots \cup I_k \notin B$.

$\mathcal{N}(B)$: the set of all nested sets of B.

Definition (fans from building sets)

First, suppose B is a connected building set on $S = \{1, \ldots, n+1\}$.

- $e_1, \ldots, e_n$: the standard basis for $\mathbb{R}^n$, $e_{n+1} = -e_1 - \cdots - e_n$.
- $e_I = \sum_{i \in I} e_i$ for $I \subset S$.
- $\mathbb{R}_{\geq 0} N = \sum_{I \in N} \mathbb{R}_{\geq 0} e_I$ for $N \in \mathcal{N}(B)$.

$\Delta(B) = \{\mathbb{R}_{\geq 0} N \mid N \in \mathcal{N}(B)\}$ is a fan in $\mathbb{R}^n$.
Thus we have an n-dimensional toric variety $X(\Delta(B))$.
If B is disconnected, then we define $X(\Delta(B)) = \prod_{C \in B_{\max}} X(\Delta(B|_C))$.

Proposition

$\Delta(B)$ is nonsingular and complete.

Remark

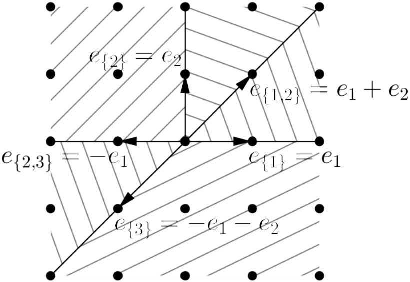
$\Delta(B)$ is the normal fan of a simple polytope called a nestohedron.
Thus $X(\Delta(B))$ is nonsingular and projective.

Example

We have $B(P_3) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$. Thus

$$\begin{aligned}\mathcal{N}(B(P_3)) = &\{\emptyset, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1,2\}\}, \{\{2,3\}\},\\ &\{\{1\},\{3\}\}, \{\{1\},\{1,2\}\}, \{\{2\},\{1,2\}\}, \{\{2\},\{2,3\}\}, \{\{3\},\{2,3\}\}\}.\end{aligned}$$

So we have the fan $\Delta(B(P_3))$ below. Therefore $X(\Delta(B(P_3)))$ is $\mathbb{P}^2$ blown-up at two points.



3. Toric Fano varieties associated to graphs

X: a nonsingular projective algebraic variety.

Definition

- X is Fano $\Leftrightarrow$ the anticanonical divisor $-K_X$ is ample.
- X is weak Fano $\Leftrightarrow$ $-K_X$ is nef and big.

X is Fano $\Rightarrow$ X is weak Fano.

G: a finite simple graph.

Theorem 1 (S)

$X(\Delta(B(G)))$ is Fano $\Leftrightarrow$ each connected component of G has ≤ 3 nodes.

Since $X(\Delta) \times X(\Delta')$ is Fano iff $X(\Delta)$ and $X(\Delta')$ are Fano, it suffices to show the following:

Theorem 1′

For a connected graph G, $X(\Delta(B(G)))$ is Fano $\Leftrightarrow |V(G)| \leq 3$.

$\Delta(r)$: the set of r-dimensional cones of Δ.

Theorem 1′

For a connected graph G, $X(\Delta(B(G)))$ is Fano $\Leftrightarrow |V(G)| \leq 3$.

Proof ($\Rightarrow$) Let $|V(G)| = n+1$, that is, $\dim X(\Delta(B(G))) = n$.

- $X(\Delta)$: n-dimensional toric Fano variety
  $\Rightarrow |\Delta(1)| \leq 3n$ (n: even), $|\Delta(1)| \leq 3n-1$ (n: odd),
  (Casagrande, 2006).
- $|\Delta(B(G))(1)| \geq |\Delta(B(P_{n+1}))(1)| = \frac{(n+1)(n+2)}{2} - 1$
  (Buchstaber–Volodin, 2011).

$\frac{(n+1)(n+2)}{2} - 1 \leq 3n$ ($\leq 3n-1$) holds only for $n \leq 2$. So $|V(G)| \leq 3$.
($\Leftarrow$) If $|V(G)| \leq 3$, then $X(\Delta(B(G)))$ must be one of the following:

- A point (G: one node).
- $\mathbb{P}^1$ ($G = P_2$).
- $\mathbb{P}^2$ blown-up at two points ($G = P_3$).
- $\mathbb{P}^2$ blown-up at three points ($G = K_3$).

Thus $X(\Delta(B(G)))$ is Fano for every case. □

G: a finite simple graph.

Theorem 2 (S)

$X(\Delta(B(G)))$ is weak Fano $\Leftrightarrow$
$\forall G'$: connected component of G and $\forall I \subsetneq V(G')$,
$G'|_I$ is neither a cycle graph of length ≥ 4 nor the diamond graph.

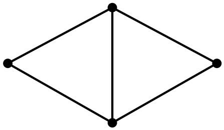
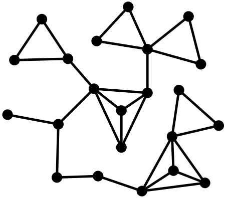
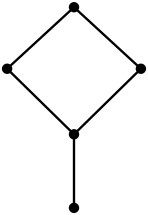


Figure: the diamond graph.

Examples

- G: a cycle graph, the diamond graph, a tree, or a complete graph $\Rightarrow X(\Delta(B(G)))$: weak Fano.
- The toric variety of the left graph is weak Fano, but the toric variety of the right graph is not weak Fano because it has a cycle graph of length 4 as a proper induced subgraph.





- Δ: a nonsingular complete fan in $\mathbb{R}^n$.
- $V(\tau)$: the torus-invariant curve corresponding to $\tau \in \Delta(n-1)$.

Proposition

Let $\tau = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_{n-1} \in \Delta(n-1)$, where $v_1, \ldots, v_{n-1}$ are primitive vectors in $\mathbb{Z}^n$, and let v and v' be the distinct primitive vectors in $\mathbb{Z}^n$ s.t. $\tau + \mathbb{R}_{\geq 0}v, \tau + \mathbb{R}_{\geq 0}v' \in \Delta(n)$.

- $\exists a_1, \ldots, a_{n-1} \in \mathbb{Z}$ s.t. $v + v' + a_1v_1 + \cdots + a_{n-1}v_{n-1} = 0$.
- The intersection number $(-K_{X(\Delta)}.V(\tau))$ is $2 + a_1 + \cdots + a_{n-1}$.

Theorem

- $X(\Delta)$ is Fano $\Leftrightarrow (-K_{X(\Delta)}.V(\tau)) > 0 \quad \forall \tau \in \Delta(n-1)$.
- $X(\Delta)$ is weak Fano $\Leftrightarrow (-K_{X(\Delta)}.V(\tau)) \geq 0 \quad \forall \tau \in \Delta(n-1)$.

- G: a connected graph with $|V(G)| = n + 1$.
- $N \in \mathcal{N}(B(G))$ with $|N| = n - 1$
  (corresponding to an $(n-1)$-dimensional cone $\mathbb{R}_{\geq 0}N$).

Key Lemma

$\exists\{J, J'\} \subset B(G) \setminus N$ s.t. $N \cup \{J\}, N \cup \{J'\} \in \mathcal{N}(B(G))$ and

$$(-K_{X(\Delta(B(G)))}.V(\mathbb{R}_{\geq 0}N)) = \begin{cases} 2 - m & (J \cup J' = V(G)), \\ 1 - m & (J \cup J' \subsetneq V(G)), \end{cases}$$

where m is the number of connected components of $G|_{J \cap J'}$.

Hence we can compute the intersection number by counting connected components of a certain induced subgraph.

Since $X(\Delta) \times X(\Delta')$ is weak Fano iff $X(\Delta)$ and $X(\Delta')$ are weak Fano, it suffices to show the following:

Theorem 2′

For a connected graph G, $X(\Delta(B(G)))$ is weak Fano $\Leftrightarrow \forall I \subsetneq V(G)$, $G|_I$ is neither a cycle graph of length ≥ 4 nor the diamond graph.

Sketch of the Proof ($\Rightarrow$) Suppose that $\exists I \subsetneq V(G)$ s.t. $G|_I$ is a cycle graph of length ≥ 4 or the diamond graph. Then we can construct $N \in \mathcal{N}(B(G))$ with $|N| = n-1$ s.t. $(-K_{X(\Delta(B(G)))}.V(\mathbb{R}_{\geq 0}N)) = -1$. Thus $X(\Delta(B(G)))$ is not weak Fano.
($\Leftarrow$) Suppose that $X(\Delta(B(G)))$ is not weak Fano. Then $\exists N \in \mathcal{N}(B(G))$ with $|N| = n-1$ s.t. $(-K_{X(\Delta(B(G)))}.V(\mathbb{R}_{\geq 0}N)) \leq -1$. By using graph-theoretic arguments, we can find $I \subsetneq V(G)$ s.t. $G|_I$ is a cycle graph of length ≥ 4 or the diamond graph. $\square$

4. Toric Fano varieties associated to buildings

Theorem 3 (S)

Let B be a building set. Then the following are equivalent:

- $X(\Delta(B))$ is Fano.
- For any $C \in B_{\max}$ and for any $I_1, I_2 \in B|_C$ s.t. $I_1 \cap I_2 \neq \emptyset, I_1 \not\subset I_2$ and $I_2 \not\subset I_1$, we have $I_1 \cup I_2 = C$ and $I_1 \cap I_2 \in B|_C$.

Remark

This implies that $X(\Delta(B(G)))$ is Fano iff each connected component of G has ≤ 3 nodes, which agrees with Theorem 1.

Problem

Find a condition for $X(\Delta(B))$ to be weak Fano in terms of the building set.

Let $G = (V(G), A(G))$ be a finite directed graph with no loops and no multiple arrows.

Definition (Higashitani)

Let $V(G) = \{1, \ldots, n+1\}$. For $\overrightarrow{e} = (i, j) \in A(G)$, we define $\rho(\overrightarrow{e}) = \mathbf{e}_i - \mathbf{e}_j \in \mathbb{R}^{n+1}$. We define P_G to be the convex hull of $\{\rho(\overrightarrow{e}) \mid \overrightarrow{e} \in A(G)\}$ in $\mathbb{R}^{n+1}$. P_G is an integral convex polytope in $H = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$.

Definition

An integral convex polytope is said to be Fano if the origin is the only lattice point in the interior, and it is said to be smooth if the vertices of every facet form a basis for the lattice.

Fact

$$\{n\text{-dim'l smooth Fano polytopes}\} \xleftrightarrow{1:1} \{n\text{-dim'l toric Fano varieties}\}.$$

Remark

Not all finite directed graphs yield smooth Fano polytopes.

Let B be a building set.

Theorem 4 (S)

If $X(\Delta(B))$ is Fano, then there exists a finite directed graph G s.t. P_G is smooth Fano and its associated fan is isomorphic to $\Delta(B)$.

dimension	1	2	3
# of toric Fano varieties	1	5	18
# of toric Fano varieties from finite directed graphs	1	5	16
# of toric Fano varieties from building sets	1	5	14
# of toric Fano varieties from finite simple graphs	1	2	3

Theorem 4 (S)

If $X(\Delta(B))$ is Fano, then there exists a finite directed graph G s.t. P_G is smooth Fano and its associated fan is isomorphic to $\Delta(B)$.

Proof By connecting finite directed graphs that yield toric Fano varieties with one node, we obtain a graph that yields a toric variety isomorphic to the product of the toric Fano varieties of the graphs. Hence it suffices to prove the assertion when B is connected.
We put $U = \{I \in B \setminus \{S\} \mid \exists J \in B \setminus \{S\} \text{ s.t. } I \cap J \neq \emptyset \text{ and } I \cup J = S\}$.

Lemma

Let B be a connected building set on S such that $X(\Delta(B))$ is Fano. Then U must be one of the following:

- $U = \emptyset$.
- $U = \{I, J\}$ for some $I, J \in B$.
- $U = \{I, J, S \setminus (I \cap J)\}$ for some $I, J \in B$.

In this talk, we prove Theorem 4 only for the case when $U = \{I, J\}$.
We may assume that $S = \{1, \ldots, n+1\}$,
$I = [1, b]$ and $J = [a, n+1]$ for some $1 < a \leq b < n+1$.
We can show that $B = \{S\} \sqcup U \sqcup B|_{I \setminus J} \sqcup B|_{I \cap J} \sqcup B|_{J \setminus I}$.
$I \setminus J, I \cap J$ and $J \setminus I$ are intervals.

Lemma

Let B be a building set on S s.t. $I, J \in B$ with $I \cap J \neq \emptyset$ implies $I \subset J$ or $J \subset I$. Then there exists a bijection $f : S \to \{1, \ldots, |S|\}$ s.t. $f(I)$ is an interval for any $I \in B$.

Since $X(\Delta(B))$ is Fano, $B|_{I \setminus J}, B|_{I \cap J}$ and $B|_{J \setminus I}$ satisfy the assumption of the Lemma. Hence we may assume that every element of B is an interval.

We may assume that every element of B is an interval.
We define a finite directed graph G as follows:
Let $V(G) = \{1, \ldots, n+1\}$. For $K = [i,j] \in B \setminus \{S\}$, we put

$$\overrightarrow{e}_K = \begin{cases} (i, j+1) & (1 \leq j \leq n), \\ (i, 1) & (j = n+1). \end{cases}$$

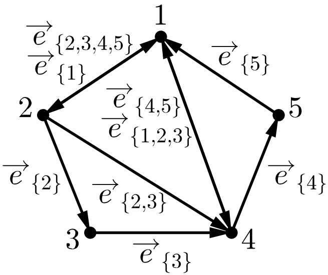
Let $A(G) = \{\overrightarrow{e}_K \mid K \in B \setminus \{S\}\}$. Then the linear isomorphism

$$F : H = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\} \to \mathbb{R}^n,$$
$$e_i - e_{i+1} \mapsto e_i$$

induces a bijection from $\{\rho(\overrightarrow{e}_K) \mid K \in B \setminus \{S\}\}$ to $\{e_K \mid K \in B \setminus \{S\}\}$, which is the set of vertices of the smooth Fano polytope corr. to $\Delta(B)$. So $P_G = \mathrm{conv}\{\rho(\overrightarrow{e}_K) \mid K \in B \setminus \{S\}\}$ is also smooth Fano. $\square$

Example

Let $S = \{1, 2, 3, 4, 5\}$ and
$B = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}, \{2, 3, 4, 5\}, \{1, \dots, 5\}\}$.
$X(\Delta(B))$ is Fano by Theorem 3.
We define a finite directed graph G by $V(G) = \{1, 2, 3, 4, 5\}$ and
$A(G) = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (2, 4), (4, 1), (1, 4), (2, 1)\}$.
Then $\Delta(B)$ is isomorphic to the fan associated to the smooth Fano polytope P_G.



Thank you for your attention!

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