The integer decomposition property and Gelfand–Tsetlin polytopes

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Motivation

Gelfand–Tsetlin polytopes appear in various areas:

- Closely related to type A_n root systems.
- Lattice points $\leftrightarrow$ semi-standard Young tableaux.
- Special faces $\leftrightarrow$ formula for Schubert polynomials
- Lattice points in some special faces $\leftrightarrow$ Demazure characters
- Contain several other families of polytopes.

Rich interplay with other combinatorial objects.

Skew Gelfand-Tsetlin patterns

A *Gelfand-Tsetlin pattern*, or GT-patterns for short, is a triangular or parallelogram arrangement of non-negative numbers,

$$\begin{array}{ccccccccccc} x_1^m & & x_2^m & & \cdots & & \cdots & & x_n^m & & \\ & \ddots & & \ddots & & & & & & \ddots & \\ & & x_1^2 & & x_2^2 & & \cdots & & \cdots & & x_n^2 \\ & & & x_1^1 & & x_2^1 & & \cdots & & \cdots & & x_n^1 \end{array}$$

where

$$\begin{array}{ccc} a & & b \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & c & \\ a & & b \end{array} \quad \Leftrightarrow \quad a \geq c \geq b.$$

A bijection

$$
\begin{array}{ccccccccccc}
5 & & 4 & & 2 & & 1 & & 1 & & 0 \\
 & 5 & & 3 & & 2 & & 1 & & 0 & \\
 & & \mathbf{3} & & \mathbf{3} & & \mathbf{2} & & \mathbf{1} & & \\
 & & & 3 & & 3 & & 1 & & & \\
 & & & & 3 & & 2 & & & & \\
 & & & & & 3 & & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|c|}
\hline
\mathbf{1} & \mathbf{1} & \mathbf{1} & 5 & 5 \\
\hline
\mathbf{2} & \mathbf{2} & \mathbf{3} & 6 \\
\cline{1-4}
\mathbf{3} & \mathbf{4} \\
\cline{1-2}
\mathbf{4} \\
\cline{1-1}
6 \\
\cline{1-1}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
4 & & 3 & & 2 & & 1 & & & & \\
 & 4 & & 3 & & 1 & & 1 & & & \\
 & & 3 & & 3 & & 1 & & 1 & & \\
 & & & 3 & & 2 & & 1 & & 0 & \\
 & & & & 2 & & 1 & & 1 & & 0
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
\cdot & \cdot & 1 & 3 \\
\cdot & 1 & 2 & \\
\cdot & 4 & & \\
2 & & &
\end{array}
$$

Gelfand-Tsetlin polytopes

Consider all GT-patterns with m rows, with top and bottom row given by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The inequalities defines a convex polytope, $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$.

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The integer points in $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ corresponds to Young tableaux with shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ and entries $\leq (m-1)$.

Marked order polytopes

The GT-polytopes are a special case of *marked order polytopes*.

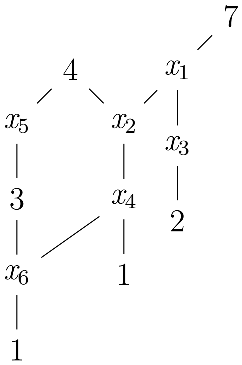

Figure : Hasse diagam of a marked poset. The inequalities we get are $1 \leq x_4 \leq x_2 \leq x_1 \leq 7$, $x_2 \leq 4$, $1 \leq x_6 \leq 3 \leq x_5 \leq 4$, $x_6 \leq x_4$ and $2 \leq x_3 \leq x_1$.

Integrally closed polytopes

A convex polytope $\mathcal{P}$ is said to have the *integer decomposition property* (IDP) if for every positive integer k and *integer* point $p \in k\mathcal{P}$, there are *integer* points $p_j \in \mathcal{P}$ such that

$$p = p_1 + p_2 + \cdots + p_k.$$

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Hence, GT-polytopes are IDP.

IDP and concatenation

Let ⊠ denote the elementwise addition of GT-patterns.

						1	1	1	1	1	5
			1	1	1	3	3	3			
1	2	2	2	2	2	4	4	5			
2	4	5									

$$=$$

		1	1
	1	3	
1	2	4	
2			

⊠

		1	1
	1	3	
2	2	4	
4			

⊠

		1	5
	1	3	
2	2	5	
5			

IDP

Generalizing:

$$(\mathcal{P}_{\boldsymbol{\lambda}_1}, \mathcal{P}_{\boldsymbol{\lambda}_2}, \dots, \mathcal{P}_{\boldsymbol{\lambda}_k})$$

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However,

$$\begin{array}{|c|}\hline ? \\ \hline ? \\ \hline ? \\ \hline\end{array} \quad \boxtimes \quad \begin{array}{|c|}\hline ? \\ \hline\end{array} = \begin{array}{cc} & 3 \\ 1 & 4 \\ 2 & \end{array}$$

has no solution, so $(\mathcal{P}_3, \mathcal{P}_{2/1})$ is not IDP.

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SAD!

Part II

Integrality and non-integrality. Motivated by research by King, Tollu, Toumazet and later de Loera and McAllister.

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Alexandersson, Per(1-PA)
Gelfand-Tsetlin polytopes and the integer decomposition property. (English summary)
European J. Combin. 54 (2016), 1–20.
17B10 (05E10 52B12)
Review PDF | Clipboard | Journal | Article | Make Link

A *Gelfand-Tsetlin polytope* is the set of all Gelfand-Tsetlin patterns with certain boundary conditions depending on three vectors λ, μ, and w. These patterns come up naturally in representation theory, where λ/μ is the *skew shape* and w is the *weight* of the patterns [I. M. Gel′fand and M. L. Tsetlin, Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 825–828; MR0035774]. Aside from representation-theoretic reasons, there are geometric-combinatorial motivations to study Gelfand-Tsetlin polytopes.

Gelfand-Tsetlin polytopes II

Let $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ be the Gelfand-Tsetlin polytope defined by the same inequalities and equalities before, *with the addition that the sum of the entries in row j resp. row $j+1$ in the pattern differ by exactly w_j.*

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\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|}
\hline
 & & 1 & 3 \\
\hline
 & 1 & 2 & \\
\hline
 & 4 & & \\
\hline
2 & & & \\
\hline
\end{array}
$$

Here, $\mathbf{w} = (2, 2, 1, 1)$ and $\mathbf{w}$ is the *type* of the tableau; w_j counts the number of boxes with content j.

Connection with structure constants

The number of integer points in $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ is equal to the number of Young tableaux with shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ and type $\mathbf{w}$.

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These numbers, denoted $K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$, are the skew Kostka numbers.

$$\begin{aligned} s_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) &= \sum_{\mathbf{w}\text{ weak integer composition}} K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} \mathbf{x}^{\mathbf{w}} \\ &= \sum_{\mathbf{w}\text{ partition}} K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} m_{\mathbf{w}}(\mathbf{x}) \end{aligned}$$

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The $K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ are special cases of Littlewood–Richardson coefficients.

Properties of $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$

The GT-polytopes $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ have strange properties.

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The GT-polytopes $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ have strange properties.

- All $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ have polynomial Ehrhart function. [1]
- Some $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ are non-integral. [2]

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Let $\overline{\mathbf{w}}$ be a permutation of the entries in $\mathbf{w}$. Then

- $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ might be integral while $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\overline{\mathbf{w}}}$ is non-integral.
- The number of integer points in $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ and $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\overline{\mathbf{w}}}$ is always the same.

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Some properties

Determining if $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ is non-empty is hard, there is no easy algorithm for this. However, all non-empty $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ contains at least one integer vertex. [3]

[3] This is a consequence of the proof of the saturation conjecture by A. Knutson and T. Tau.

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The case $|\boldsymbol{\mu}| = 0$ (non-skew case) is easy, there is a simple condition on $\boldsymbol{\lambda}$ and $\mathbf{w}$ which is necessary and sufficient.

[3] This is a consequence of the proof of the saturation conjecture by A. Knutson and T. Tau.

Methods

The main resource is Jesús A. De Loera and Tyrrell B. McAllister, *Vertices of Gelfand-Tsetlin Polytopes*, Discrete & Computational Geometry **32** (2004), no. 4, 459–470.

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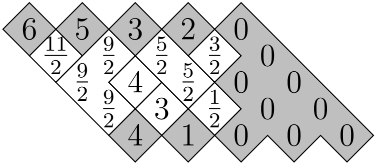
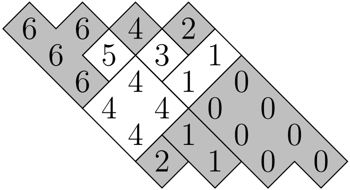
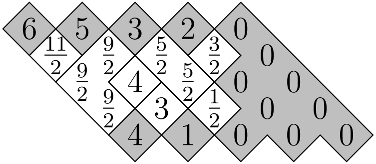
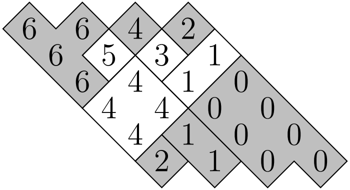




Methods

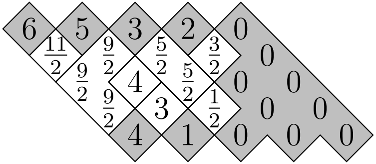


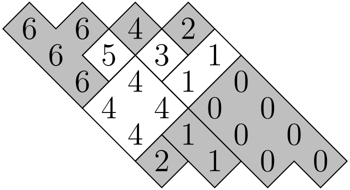


$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Methods





$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem: The dimension of ker T_G is equal to the dimension of the minimal (dimensional) face of the GT-polytope containing G.

Main results

Theorem (A. 2014)

All *integral* $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$ are *compressed*.

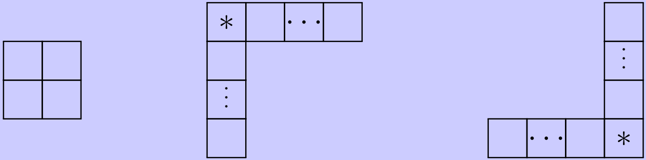
This implies the existence of a unimodular triangulation and IDP.

Note that the polytope $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$ is always non-empty and that integer points in this polytope correspond to *standard* Young tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$.

INTEGRALITY OF $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$

Theorem (A. 2014)

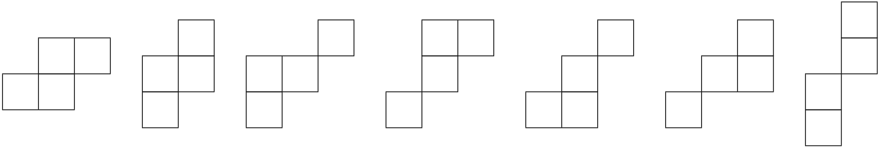
The only shapes $\boldsymbol{\lambda}/\boldsymbol{\mu}$ for which $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{1}}$ is integral are

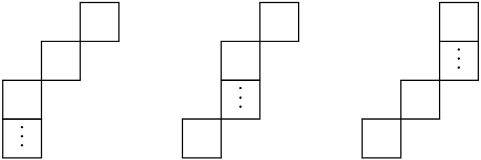


or a disjoint union of rows of boxes.

Proof idea

- Find subdiagram patterns that admit a non-integral vertex.
- Prove that all skew diagrams that avoid these patterns must be of the form above.





(four or more boxes)

Refinement results

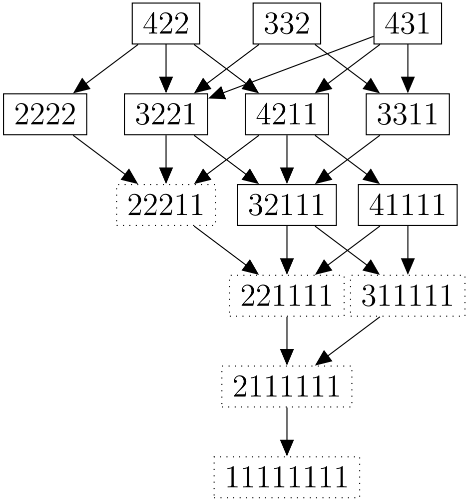
Proposition (A. 2014)

Suppose $\mathbf{w}' <_{\text{ref}} \mathbf{w}$ and let $P = \mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ and $P' = \mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}'}$. Then

1. $|P' \cap \mathbb{Z}^{d'}|$ is greater or equal to $|P \cap \mathbb{Z}^{d}|$. (Easy)
2. If P' is integral, then P is integral. (GT-proof)
3. If P' is integrally closed, then so is P. (Tableau-proof)

Example

Non-skew case $\boldsymbol{\lambda} = 431$, and $\mathbf{w}$ in the boxes.



Note: only partitions $\mathbf{w}$ are shown here.

Part III

Related questions and big counterexamples.

Ehrhart polynomial: $p(k) = |k\mathcal{P} \cap \mathbb{Z}^d|$.

Leading coefficient = volume of the polytope.

Open problem

Conjecture (King, Tollu Toumazet, 2004)

All coefficient in the Ehrhart polynomial obtained from $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ are non-negative.

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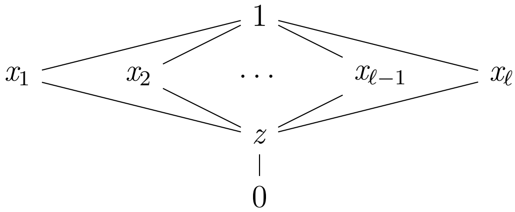
All coefficient in the Ehrhart polynomial obtained from $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ are non-negative.

This conjecture contains the problem about positivity of Ehrhart polynomial of the Birkhoff polytope.

Maybe the conjecture is false?

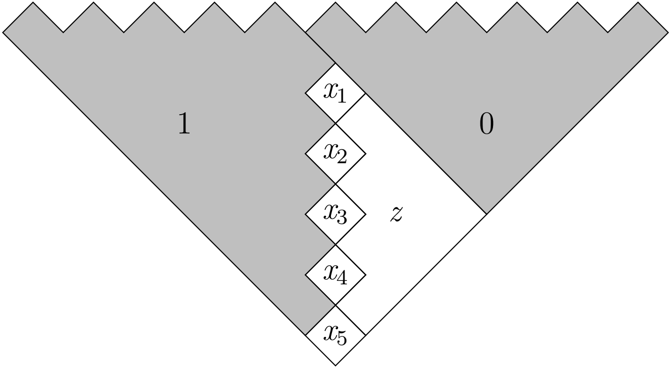
There are order polytopes with negative Ehrhart coefficients:



This order polytope has Ehrhart polynomial given by $p(k) = \sum_{j=1}^{k+1} j^{\ell}$. For $\ell = 20$, there are negative coefficients.

Negative coefficients

Consequence: There are Ehrhart polynomials of *faces* of $\mathcal{P}_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ with *negative* coefficients:



The end

Thank you for your time