How connected is the skeleton of a polytope?

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Introduction

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57M15 (05E45 13F55 55U10 57Q05)

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Citations
From References: 0
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Summary: "In this note we generalize and unify two results on connectivity of graphs: one by Balinsky and Barnette, one by Athanasiadis. This is done through a simple proof using commutative algebra tools. In particular we use bounds for the Castelnuovo-Mumford regularity of their Stanley-Reisner rings. As a result, if Δ is a simplicial d-pseudomanifold and s is the largest integer such that Δ has a missing face of size s, then the 1-skeleton of Δ is $\lceil \frac{(s+1)d}{s} \rceil$-connected. We also show that this value is tight."

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The notion of connectivity

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We say that a graph G having more than m vertices is m-connected whenever it is impossible to disconnect it by removing fewer than m vertices together with their incident edges.

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Facts:

- Any connected graph G is 1-connected.
- Any tree T is 1-connected but not 2-connected.
- The cycle graph C_n is 2-connected.
- The complete bipartite graph $K_{m,n}$ is $(\min\{m, n\})$-connected.
- The complete graph K_n is $(n-1)$-connected

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A graph G is the skeleton of a 3-dimensional polytope if and only if G is a 3-connected planar graph.

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Theorem (Balinski, 1961)

The skeleton of a d-dimensional polytope is d-connected.

Simplicial complexes

Definition

A *simplicial complex* Δ on the vertex set $[n] := \{1, \ldots, n\}$ is a collection of subsets (faces) of $[n]$ such that such that

- $\varnothing \in \Delta$,
- $F \in \Delta$ and $G \subseteq F \Rightarrow G \in \Delta$.

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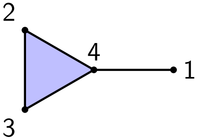

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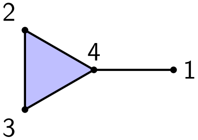

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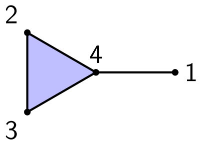

We call Δ **pure** if all the maximal faces (facets) have the same dimension.
We can think to a $(d+1)$-dimensional simplicial polytope P as the pure d-dimensional simplicial complex whose faces are the proper faces of P.

Pseudomanifolds

Definition

A pure d-dimensional simplicial complex Δ is a **simplicial pseudomanifold** if

- every $(d-1)$-dimensional faces is contained in exactly two facets,
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Theorem (Barnette, 1982)

The skeleton of a d-dimensional simplicial pseudomanifold is $(d+1)$-connected.

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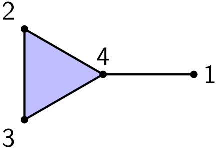

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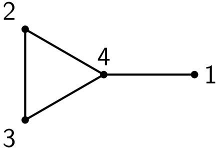

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Theorem (Athanasiadis, 2011)

The skeleton of a flag simplicial d-pseudomanifold is $2d$-connected.

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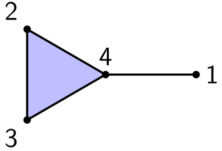

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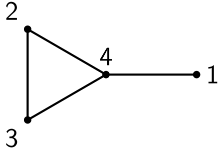

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Goal: Prove (and maybe improve) the results by Barnette and Athanasiadis.

Let's use commutative algebra!

Let Δ be a simplicial complex on n vertices.
Let $\mathbb{k}$ be any field and S the polynomial ring $S = \mathbb{k}[x_1, \ldots, x_n]$.

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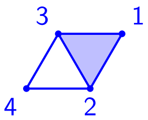
Example: Let $S = \mathbb{k}[x_1, x_2, x_3, x_4]$



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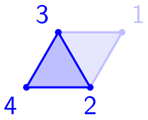

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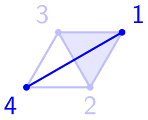

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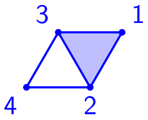

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It is enough to use the minimal nonfaces.

Let's do algebra: free resolution

Let M be a finitely generated graded S-module. Hilbert's Syzygy Theorem grants the existence of a **minimal graded free resolution** of M, i.e. a chain complex $\mathbf{F}$ of graduated free modules of minimal rank with degree-preserving maps such that $\mathbf{F} \to M \to 0$ is exact.

$$0 \to \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{s,j}} \xrightarrow{\phi_s} \cdots \xrightarrow{\phi_2} \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \xrightarrow{\phi_1} \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{0,j}} \to 0$$

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We store the Betti numbers in the **Betti table**

$$\beta(M) = \begin{bmatrix} \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots & \beta_{s,s} \\ \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{s,s+1} \\ \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \cdots & \beta_{s,s+2} \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

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$\mathrm{projdim}(M) = \#$ columns of $\beta(M) - 1$ $\qquad$ $\mathrm{reg}(M) = \#$ rows of $\beta(M)$

Hochster Formula

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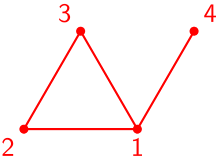
Theorem (Hochster, 1977)

The Betti numbers of $\Bbbk[\Delta]$ *can be calculated studying the homology of* Δ*:*

$$\beta_{i,j}(\Bbbk[\Delta]) = \sum_{\substack{T \subseteq [n] \\ |T| = j}} \dim_{\Bbbk} \widetilde{H}_{j-i-1}(\Delta|_T; \Bbbk).$$

- The **reduced homology** $\widetilde{H}_i(\,\cdot\,; \Bbbk)$ is like the usual homology, but in degree 0 it "counts" the connected components minus one.
- $\Delta|_T$ denotes the restriction of Δ to T.

Example: the algebraic way



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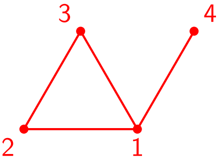

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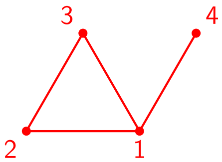

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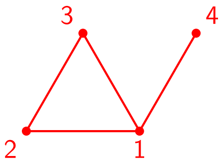

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Example: the combinatorial way

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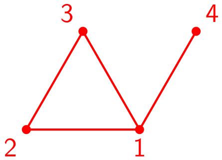

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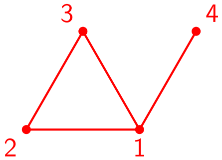

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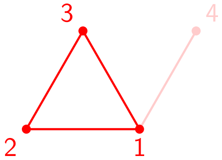

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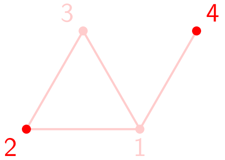

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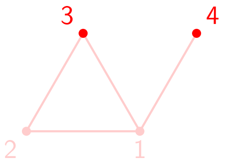

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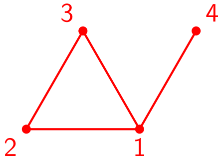

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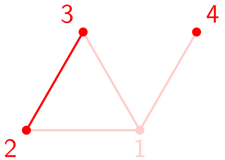

$$T = \{2,3,4\} \Longrightarrow \dim_{\Bbbk} \widetilde{H}_0(\Delta|_T; \Bbbk) = 1 \Longrightarrow \beta_{2,3} \to +1$$

$$\beta(\Bbbk[\Delta]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The Castelnuovo-Mumford regularity

$$\mathrm{reg}(\Bbbk[\Delta]) = \#\text{ rows of } \beta(\Bbbk[\Delta])$$

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$$\beta_{i,j}(\Bbbk[\Delta]) = \sum_{\substack{T \subseteq [n] \\ |T| = j}} \dim_{\Bbbk} \widetilde{H}_{j-i-1}(\Delta|_T; \Bbbk).$$

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- In every row the homological degree is fixed.
- The Castelnuovo-Mumford regularity equals the highest homological degree with nontrivial homology.

Lemma

Let Δ be a d-dimensional simplicial pseudomanifold and let $T \subseteq \Delta$ such that $\Delta|_{[n]\setminus T}$ is disconnected. Then $\mathrm{reg}(\Bbbk[\Delta|_T]) \geq d$.

Proof: use Hochster Formula and Mayer-Vietoris sequence.

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Let $I \subseteq S$ a monomial ideal generated by monomials of degree at most s. Then

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$$N \geq \frac{s}{s-1}\mathrm{reg}(\Bbbk[\Delta|_T]) \geq \frac{sd}{s-1}$$

Main theorem

Theorem (B., 2015)

Let Δ be a d-dimensional simplicial pseudomanifold. Let s be the dimension of the largest minimal non-face of Δ. Then the skeleton G_Δ of Δ is $\lceil \frac{sd}{s-1} \rceil$-connected.

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It works under weaker hypotheses: we do not need Δ to be a d-dimensional pseudomanifold, but just to be a **vertex minimal d-cycle**, i.e. for some field $\mathbb{k}$, $\widetilde{H}_d(\Delta|_T; \mathbb{k}) \neq 0$ if and only if $T = [n]$.

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It is a sharp result (simplicial join are the extremal objects)

We can keep having fun!

Theorem (Dao-Huneke-Schweig, 2013)

Let $I \subseteq S$ a monomial ideal generated in degree 2 such that I is k-step linear. Then

$$\mathrm{reg}(S/I) \leq \min\left\{\log_{\frac{k+4}{2}}\left(\frac{n-1}{k+1}\right)+2, \log_{\frac{k+4}{2}}\left(\frac{(n-1)\ln(\frac{k+4}{2})}{k+1}+\frac{2}{k+4}\right)+2\right\}.$$

$$\beta(I) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{k,k+1} & \beta_{k+1,k+2} & \cdots \\ 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \beta_{k+1,k+3} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \end{bmatrix},$$

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The bold zeroes corresponds (by Hochster Formula) to homological degree one → We are just counting cycles!

Theorem (B., 2015)

Let Δ be a d-dimensional simplicial pseudomanifold such that its skeleton G_Δ has no induced k-cycles for $k \geq 4$. Then G_Δ is $\lceil (\frac{k}{2})^{d-1} \rceil$-connected.

Find your own result on connectivity

- Take a good bound for the regularity

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- Cite my paper!!

Thank you!