Lattice 3-polytopes: quantum jumps and interior points

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July 31, 2018

Summer Workshop on Lattice Polytopes
University of Osaka, Japan

LATTICE DISTANCE(s)

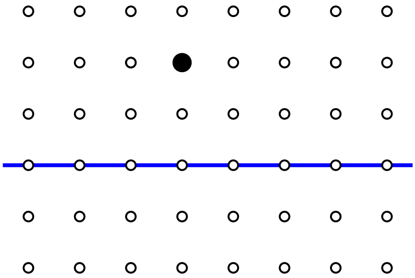
Lattice distance between a *point* to a *hyperplane*

- Our ambient lattice will always be $\mathbb{Z}^d$.
- **Lattice polytope** $P :=$ convex hull of a finite set of points in $\mathbb{Z}^d$.
- In this talk, P is always a lattice polytope in $\mathbb{R}^d$.
- $f : \mathbb{R}^d \to \mathbb{R}$ affine *integer* functional. It is primitive if $f(\mathbb{Z}^d) = \mathbb{Z}$.

- The **lattice distance** between point $x \in \mathbb{Z}^d$ and lattice hyperplane $H \subset \mathbb{R}^d$ is

$$\mathrm{dist}(x, H) = |f(x)|$$

where f is a primitive functional with $f(H) = 0$.



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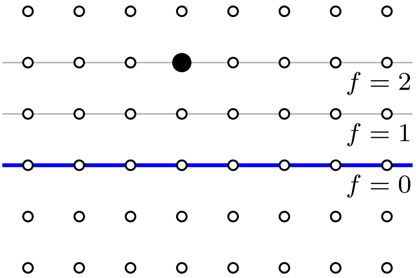

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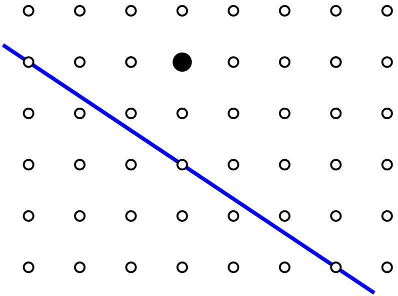

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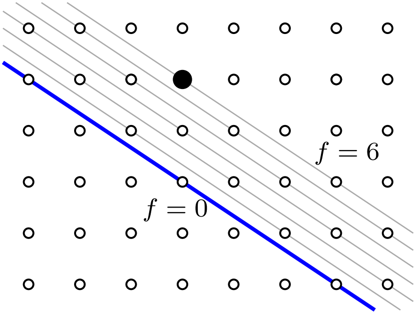
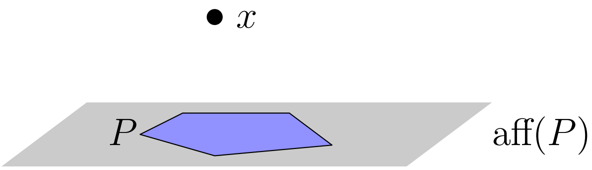


Distances between . . . and . . .

- point and $(d-1)$-dim polytope: $P \subset \mathbb{R}^d$, $dim(P) = d-1$; $x \in \mathbb{Z}^d \setminus \text{aff}(P)$:

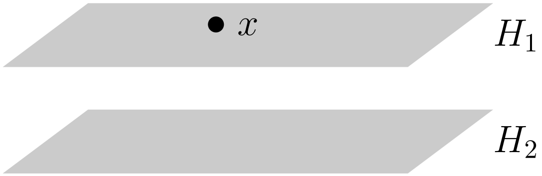
$$\text{dist}(x, P) := \text{dist}(x, \text{aff}(P))$$



Distances between . . . and . . .

- lattice hyperplanes: $H_1, H_2 \subset \mathbb{R}^d$ lattice hyperplanes, parallel ($H_1 \cap H_2 = \emptyset$):

$$\operatorname{dist}(H_1, H_2) := \operatorname{dist}(x, H_2), \text{ for any } x \in H_1$$

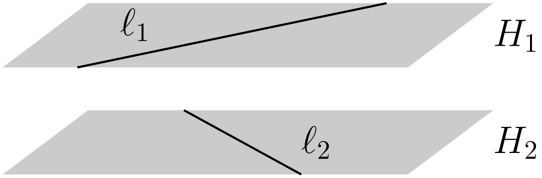


Distances between . . . and . . .

- lattice lines in $\mathbb{R}^3$: let $\ell_1, \ell_2 \subset \mathbb{R}^3$ lattice lines s. t. $\mathrm{aff}(\mathrm{conv}(\ell_1 \cup \ell_2)) = \mathbb{R}^3$:

$$\mathrm{dist}(\ell_1, \ell_2) := \mathrm{dist}(H_1, H_2)$$

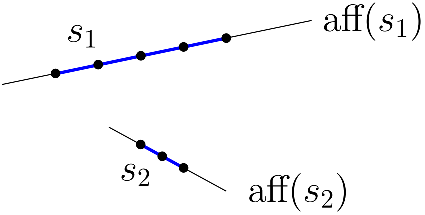
where H_1, H_2 are the unique pair of parallel lattice hyperplanes such that $\ell_i \subset H_i$.



Distances between ... and ...

- lattice segments in $\mathbb{R}^3$: let $s_1, s_2 \subset \mathbb{R}^3$ lattice segments s. t. $\mathrm{aff}(\mathrm{conv}(s_1 \cup s_2)) = \mathbb{R}^3$:

$$\mathrm{dist}(s_1, s_2) := \mathrm{dist}\left(\mathrm{aff}(s_1), \mathrm{aff}(s_2)\right)$$



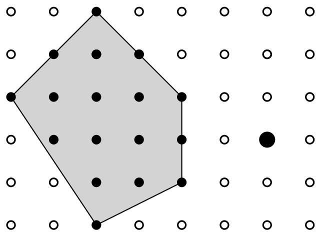
Distances between ... and ...

- point and $(d-1)$-dim polytope
- lattice hyperplanes
- lattice lines in $\mathbb{R}^3$
- lattice segments in $\mathbb{R}^3$
- ...

In general, if $R := \mathrm{conv}(P \cup Q) \subset \mathbb{R}^d$ is not full-dim, the distance is measured in the lattice $\mathrm{aff}(R) \cap \mathbb{Z}^d \cong \mathbb{Z}^{dim(R)}$

Distance to full-dim polytopes

$P \subset \mathbb{R}^d$ full-dimensional, $x \in \mathbb{Z}^d$
$x \notin P \implies \text{dist}(x, P)$???



Definition

F facet of P is **visible** from x if $\text{aff}(F)$ strictly separates x from P.

We consider two different *distances*:

- The *minimum facet distance*
  $d_x(P) := \min\{\text{dist}(x, \text{aff}(F)) \mid F \text{ facet visible from } x\}$
- The *maximum facet distance*
  $D_x(P) := \max\{\text{dist}(x, \text{aff}(F)) \mid F \text{ facet visible from } x\}$

In example, $d_x(P) = 1$ and $D_x(P) = 2$.

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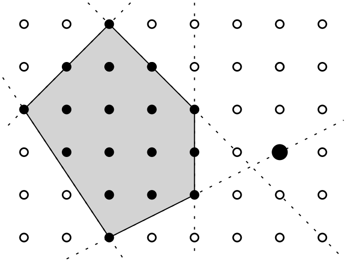

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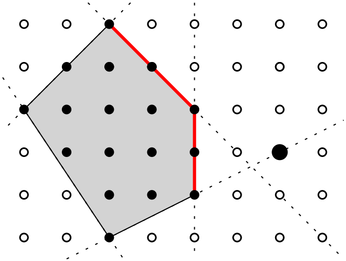

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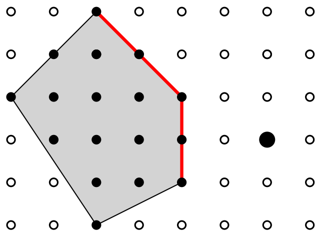

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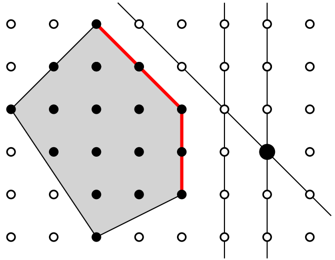

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Quantum jumps and unions

Quantum jumps & unions

Definition

- $P \subset \mathbb{R}^d$ not necessarilly full-dimensional, $x \in \mathbb{Z}^d$. We say that the pair (P, x) is a **quantum jump** if

$$\operatorname{conv}(P \cup \{x\}) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) \cup \{x\}$$

- $P, Q \subset \mathbb{R}^d$ not necessarilly full-dimensional. We say that the pair (P, Q) is a **quantum union** if

$$\operatorname{conv}(P \cup Q) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) \cup (Q \cap \mathbb{Z}^d)$$

(That is, if $p \in \operatorname{conv}(P \cup Q) \cap \mathbb{Z}^d$, then $p \in P$ or $p \in Q$)

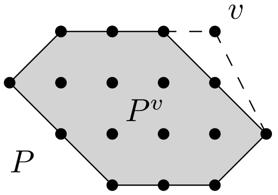
Quantum "=" convex hull does not add more lattice points

Why the name: Used by Bruns–Gubeladze–Michałek (with slight differences) ... but mainly because it sounds cool!!!

Quantum jumps distances

Let $P \subset \mathbb{R}^d$ be a lattice d-polytope and let $v \in \text{vert}(P)$. Denote

$$P^v := \text{conv}(P \setminus \{v\} \cap \mathbb{Z}^d) \subset \mathbb{R}^d$$



We study, for each dimension d:

$$\left\{ \text{dist}(v, P^v) \mid P \text{ lattice } d\text{-polytope}, \ v \in \text{vert}(P) \right\}$$

WHY??

Plenty of information already from previous research (classification of lattice 3-polytopes with small number of lattice points)...

APPLICATIONS???

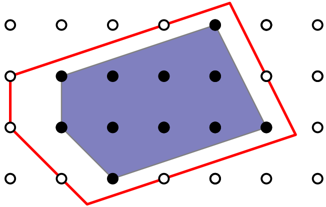
Quantum jumps distances DIM 1 and 2

$\underline{d=1}$: $(\bullet, \bullet)$ is quantum jump $\iff$ dist$(\bullet, \bullet) = 1$ in $\mathbb{R}$

$$\left\{ \text{dist}(v, P^v) \mid P \text{ lattice } \underline{\text{segment}},\ v \in \text{vert}(P) \right\} = \{1\}$$

$\underline{d=2}$: $\left(\bullet, /\right)$ is quantum jump $\iff$ dist$\left(\bullet, /\right) = 1$ in $\mathbb{R}^2$

$$\left\{ \text{dist}(v, P^v) \mid P \text{ lattice } \underline{\text{polygon}},\ v \in \text{vert}(P) \right\} = \{1\}$$



Quantum jumps distances DIM 3

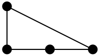
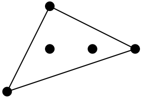
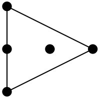
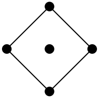
What about dimension 3???

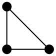
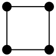
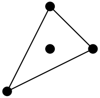
$$\left\{ \mathrm{dist}(v, P^v) \mid P \text{ lattice 3-polytope}, \ v \in \mathrm{vert}(P) \right\} = ???$$

- **"2dim to 3dim":** Distances $\mathrm{dist}(v, P^v)$ for the quantum jump (v, P^v), when P^v is 2-dimensional.
- **"3dim to 3dim":** Distances $\mathrm{dist}(v, P^v)$ for the quantum jump (v, P^v), when P^v is 3-dimensional.

Quantum jumps distances *"2dim to 3dim"*

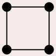
Previous research:

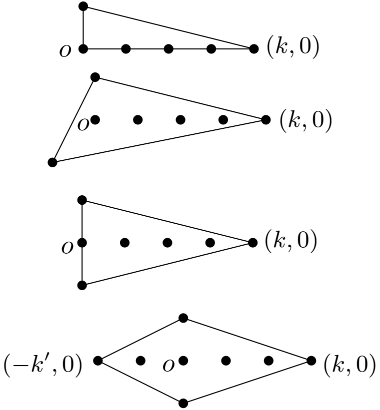
quantum jump		distance
	∪ •	(*)
	∪ •	1
	∪ •	1 or 2
	∪ •	1 or 2

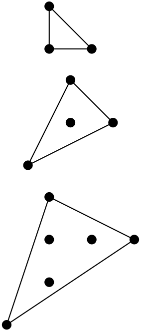
quantum jump		distance
	∪ •	(*)
	∪ •	1
	∪ •	1 or 3

(*) Unbounded

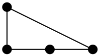
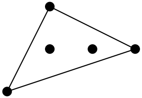
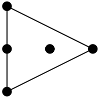
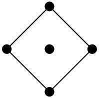
Lemma

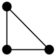
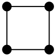
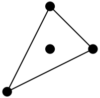
Let $Q \subset \mathbb{R}^2$ be a lattice polygon. Then Q contains  *or is equal to:*





Previous research:

quantum jump		distance
	∪ •	(*)
	∪ •	1
	∪ •	1 or 2
	∪ •	1 or 2

quantum jump		distance
	∪ •	(*)
	∪ •	1
	∪ •	1 or 3

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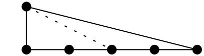
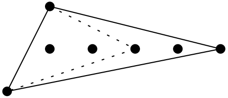
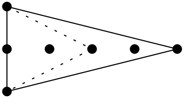
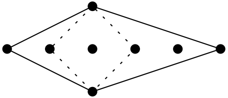
Previous research.... EXTENDED

quantum jump		distance
	$\cup$ •	(*)
	$\cup$ •	1
	$\cup$ •	1 or 2
	$\cup$ •	1 or 2

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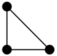
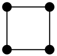
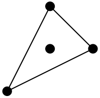
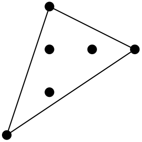
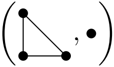
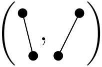
	$\cup$ •	(*)
	$\cup$ •	1
	$\cup$ •	1 or 3
	$\cup$ •	1 or 2

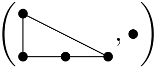
(*) Unbounded... but distances bounded for quantum unions...

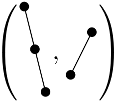
EXTRA: distance of *quantum union*

(*) Unbounded cases:

-  quantum jump $\Longleftrightarrow$ empty tetrahedron $\Longleftrightarrow$

  $\Longleftrightarrow$  quantum union at distance 1 (White, '64)

-  quantum jump $\Longleftrightarrow$

  $\Longleftrightarrow$  quantum union at distance 1 (B.–Santos, '16)

Lemma

Let $s_1, s_2 \subset \mathbb{R}^3$ lattice segments, at least one of them non-primitive

$$(s_1, s_2) \textit{ is a quantum union} \Longleftrightarrow \operatorname{dist}(s_1, s_2) = 1$$

Quantum jumps *"3dim to 3dim"*

DATABASE: P lattice 3-polytope of width > 1 and ≤ 11 lattice points. $216,453$

- $(D_v(P^v))_{v \in \text{vert}(P),\ P^v \text{ full-dim}} =: \overline{D}_P$
- $(d_v(P^v))_{v \in \text{vert}(P),\ P^v \text{ full-dim}} =: \overline{d}_P$

Three cases:

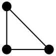
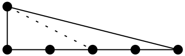
BEST $\overline{d}_P = (1, 1, \ldots, 1)$, $\overline{D}_P = (1, 1, \ldots, 1)$ $5,796$
(any vertex v of P is at distance 1 from all the visible facets of P^v)

MEH... $\overline{d}_P = (1, 1, \ldots, 1)$, $\overline{D}_P \neq (1, 1, \ldots, 1)$ $77,443$
(any vertex v of P is at distance 1 from at least one of the visible facets of P^v)

WORST $\overline{d}_P \neq (1, 1, \ldots, 1)$, $\overline{D}_P \neq (1, 1, \ldots, 1)$ $133,214$
(a vertex v of P is at distance > 1 from all the visible facets of P^v)

Moreover, the highest values in the $\overline{d}_P$'s and the $\overline{D}_P$'s are **37** and **43**, respectively.

WHY??

Recall: if for some v the only visible facet of P^v from v is  or , the distance is $d_v(P^v) = D_v(P^v)$ unbounded!!!

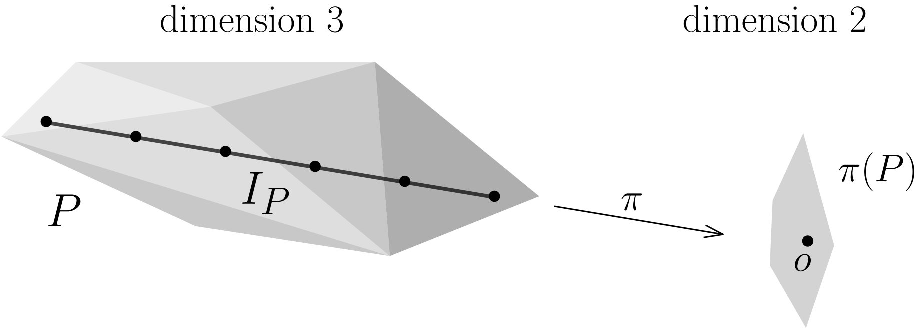
$$\left\{ \mathrm{dist}(v, P^v) \mid P \text{3-dim}, \; v \in \mathrm{vert}(P), P^v \text{ full-dim} \right\} = \mathbb{N}$$

Let's go back in time...

Once upon a time in Oberwolfach

Theorem (Averkov–Balletti–B.–Nill–Soprunov, Sep. 2017)

Let $P \subseteq \mathbb{R}^3$ be a lattice 3-polytope with $I_P := \operatorname{int}(P) \cap \mathbb{Z}^3$ a lattice segment of lattice length $k > 0$ ($k+1$ collinear lattice points).
Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the lattice projection that maps I_P to the origin.
If $k \geq 2$, $\pi(P)$ is a reflexive polygon.



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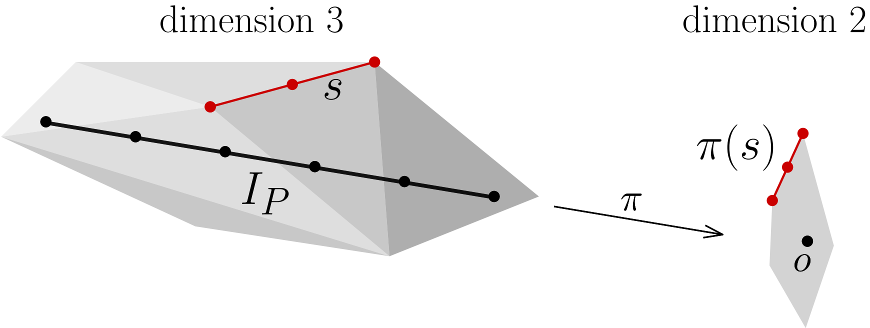

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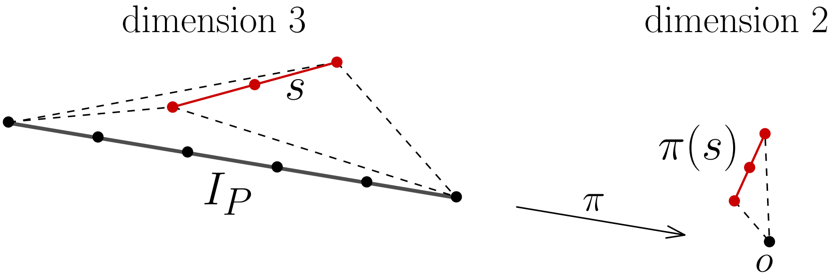

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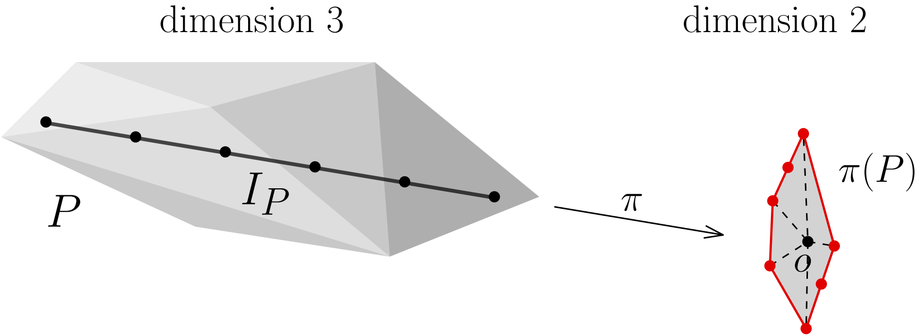
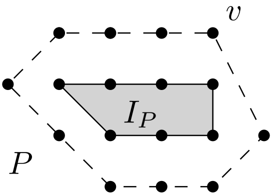


Quantum "inner" jumps distances

Let $P \subset \mathbb{R}^d$ be a lattice d-polytope, denote

$$I_P := \operatorname{conv}\left(\operatorname{int}(P) \cap \mathbb{Z}^d\right) \subset \mathbb{R}^d$$

the **inner polytope** of P.

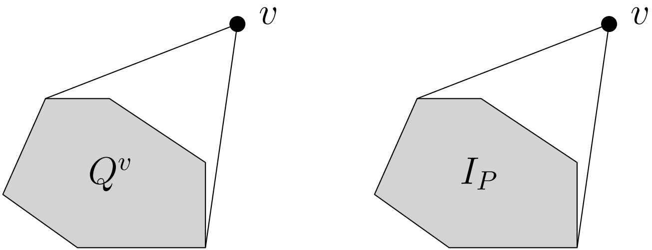


We study, for each dimension d:

$$\left\{ \operatorname{dist}(v, I_P) \mid P \text{ lattice } d\text{-polytope, } v \in \operatorname{vert}(P) \right\}$$

Why is this better?

Why do we expect to get smaller distances?



I_P closed subset of $\text{int}(P) \implies \exists \epsilon > 0$ such that $I_P + \epsilon \mathcal{B}_d \subset \text{int}(P)$
Then: (v, I_P) quantum jump $\iff (v, I_P + \epsilon\mathcal{B}_d)$ quantum jump.

That is, there are **EXTRA RESTRICTIONS** hidden in the fact that we are looking at the distance with the inner polytope.

(Also for quantum unions)

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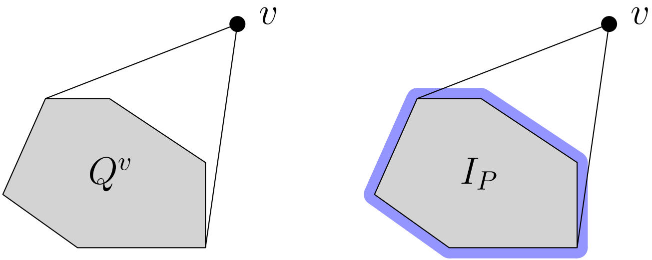

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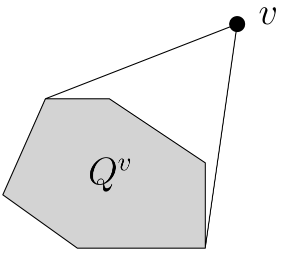


I_P closed subset of $\text{int}(P) \Longrightarrow \exists \epsilon > 0$ such that $I_P + \epsilon \mathcal{B}_d \subset \text{int}(P)$
Then: (v, I_P) quantum jump $\Longleftrightarrow$ $(v, I_P + \epsilon \mathcal{B}_d)$ quantum jump.

That is, there are **EXTRA RESTRICTIONS** hidden in the fact that we are looking at the distance with the inner polytope.

(Also for quantum unions)

Quantum "inner" jumps distances DIM 1 and 2

AGAIN: (it cannot be more restrictive than distance 1)

$d = 1$:

$$\left\{ \mathrm{dist}(v, I_P) \mid P \text{ lattice } \underline{\text{segment}},\ v \in \mathrm{vert}(P) \right\} = \{1\}$$

$d = 2$:

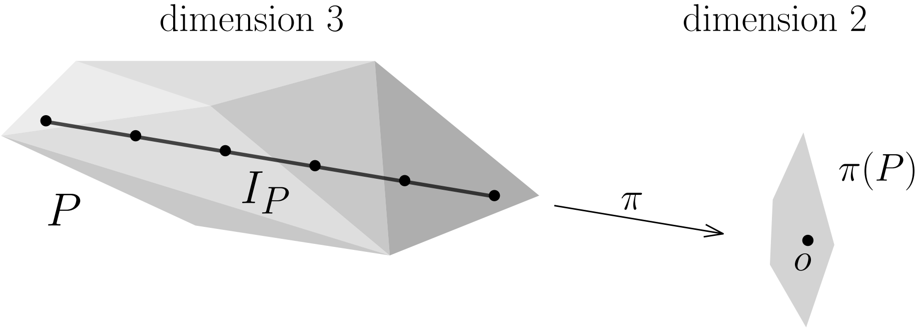
$$\left\{ \mathrm{dist}(v, I_P) \mid P \text{ lattice } \underline{\text{polygon}},\ v \in \mathrm{vert}(P) \right\} = \{1\}$$

Quantum “inner” jumps distances DIM 3

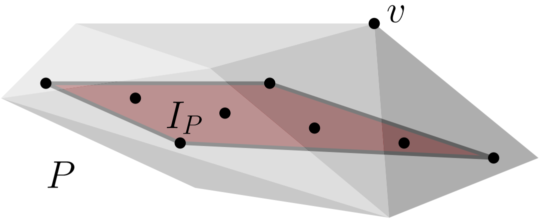
- 1 interior lattice point $\Longrightarrow$ classification by Kasprzyk
- 2 interior lattice points $\Longrightarrow$ classification by Balletti & Kasprzyk
- ≥ 3 collinear interior lattice points $\Longrightarrow$ **reflexive projection**
- 2-dimensional inner polytope $\Longrightarrow$ (*)
- 3-dimensional inner polytope $\Longrightarrow$ (*)

1-dimensional inner polytope

I_P 1-dimensional $\Longrightarrow$ reflexive projection

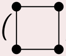


2-dimensional inner polytope



Theorem

Let P be a lattice 3-polytope of size ≥ 4 such that I_P is 2-dimensional. Then:

- *If I_P contains a unit square (*  *), then* $\mathrm{dist}(v, I_P) \leq 1$, $\forall v \in \mathrm{vert}(P)$.
- *If I_P DOES NOT contain a unit square, then* $|I_P \cap \mathbb{Z}^3| \leq 11$.

3-dimensional inner polytope

DATABASE: P lattice 3-polytope with ≤ 11 lattice points and 3-dimensional I_P. $15,763$

- $(D_v(I_P))_{v\in \text{vert}(P)} =: \overline{D}^I_P$
- $(d_v(I_P))_{v\in \text{vert}(P)} =: \overline{d}^I_P$

Three cases:

BEST $\overline{d}^I_P = (1,1,\dots,1)$, $\overline{D}^I_P = (1,1,\dots,1)$ $8,786$

(any vertex v of P is at distance 1 from all the visible facets of I_P)

MEH... $\overline{d}^I_P = (1,1,\dots,1)$, $\overline{D}^I_P \neq (1,1,\dots,1)$ $5,804$

(any vertex v of P is at distance 1 from at least one of the visible facets of I_P)

WORST $\overline{d}^I_P \neq (1,1,\dots,1)$, $\overline{D}^I_P \neq (1,1,\dots,1)$ $1,173$

(a vertex v of P is at distance ≥ 1 from all the visible facets of I_P)

Also, the highest values in the $\overline{d}^I_P$'s and the $\overline{D}^I_P$'s are **4** and **6**, respectively.

Summary of information

For polytopes with interior lattice points:

- 1 interior lattice point $\Longrightarrow$ classification by Kasprzyk
- 2 interior lattice points $\Longrightarrow$ classification by Balletti & Kasprzyk
- ≥ 3 collinear interior lattice points $\Longrightarrow$ **reflexive projection**
- 2-dimensional inner polytope $\Longrightarrow$ **distance** 1 **or few interior lattice points** TO COMPLETE
- 3-dimensional inner polytope $\Longrightarrow$ **MAYBE any vertex is at distance at most** 4 **from the inner polytope** TO COMPLETE

Thank you for your attention!!