Christopher Borger

The mixed volume and the mixed degree of a family of lattice polytopes

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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Outline

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The lattice mixed volume

Definition

Let $P_1, \ldots, P_n \subset \mathbb{R}^n$ lattice polytopes. Define *lattice mixed volume* by

$$\mathrm{MV}(P_1, \ldots, P_n) := \sum_{I \subseteq [n]} (-1)^{n-|I|} \mathrm{Vol}_n(P_I),$$

where $P_I := \sum_{i \in I} P_i$ is the *Minkowski sum* of all P_i with $i \in I$.

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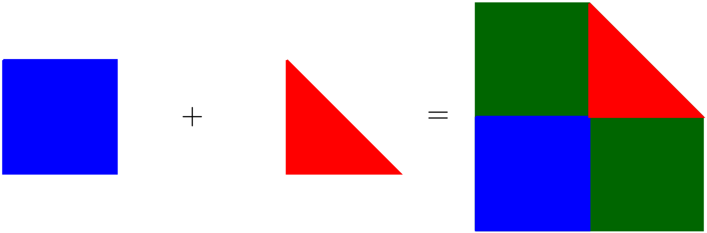
Example

Let $P_1, P_2 \in \mathbb{R}^2$ lattice polytopes, then

$$\mathrm{MV}(P_1, P_2) = \mathrm{Vol}_n(P_1 + P_2) - \mathrm{Vol}_n(P_1) - \mathrm{Vol}_n(P_2).$$



Properties of the lattice mixed volume see e.g. [Sch14]

- $\mathrm{MV}(P_1, \ldots, P_n) \in \mathbb{Z}_{\geqslant 0}$ and for full-dimensional polytopes $\mathrm{MV}(P_1, \ldots, P_n) \geqslant 1$

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- MV is multilinear, that is

$$\begin{aligned}\mathrm{MV}(\lambda P_1 + \mu P_1', P_2, \ldots, P_n) = &\lambda\,\mathrm{MV}(P_1, P_2, \ldots, P_n) \\ &+\mu\,\mathrm{MV}(P_1', P_2, \ldots, P_n).\end{aligned}$$

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- MV is symmetric

Algebro-geometric meaning

Theorem (B(ernstein)K(hovanskii)K(ushnirenko))

Let $f_1, \dots, f_n \in \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ generic Laurent polynomials with Newton polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$.

$$\#\{\textit{roots of } f_1, \dots, f_n \textit{ in } (\mathbb{C}^*)^n\} = \mathrm{MV}(P_1, \dots, P_n)$$

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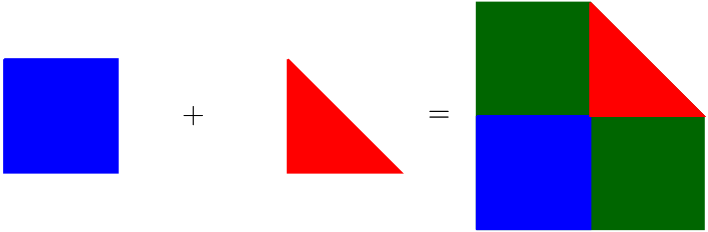

$a_0 + a_1x + a_2y + a_3xy$ $\qquad$ $b_0 + b_1x + b_2y$

Outline

Mixed volume constellations

Fix full-dimensional lattice polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$.
For $x_1, \ldots, x_n \in \mathbb{Z}_{\geqslant 0}$ such that $x_1 + \cdots + x_n = n$, denote

$$\begin{aligned}\mathrm{MV}(x_1, \ldots, x_n) &:= \mathrm{MV}(\underbrace{P_1, \ldots, P_1}_{x_1 \text{ times}}, \ldots, \underbrace{P_n, \ldots, P_n}_{x_n \text{ times}}) \\ &= \mathrm{MV}(P_1[x_1], \ldots, P_n[x_n]).\end{aligned}$$

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E.g., given $P_1, P_2, P_3 \subset \mathbb{R}^3$ write

- $\mathrm{MV}(1,1,1) = \mathrm{MV}(P_1, P_2, P_3)$
- $\mathrm{MV}(2,1,0) = \mathrm{MV}(P_1, P_1, P_2)$
- $\mathrm{MV}(0,0,3) = \mathrm{MV}(P_3, P_3, P_3) = \mathrm{Vol}_{\mathbb{Z}^n}(P_3)$

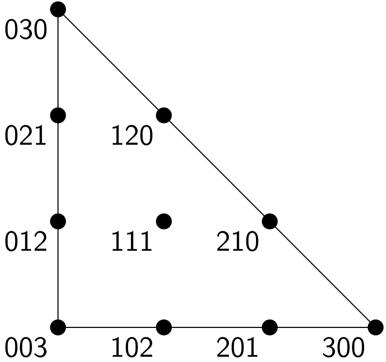
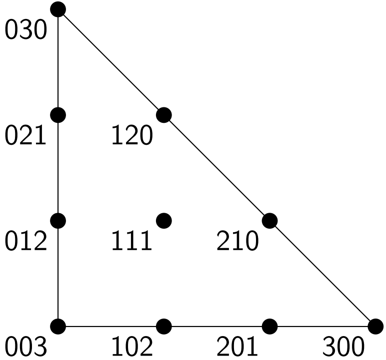


Figure: The mixed volumes for $n = 3$



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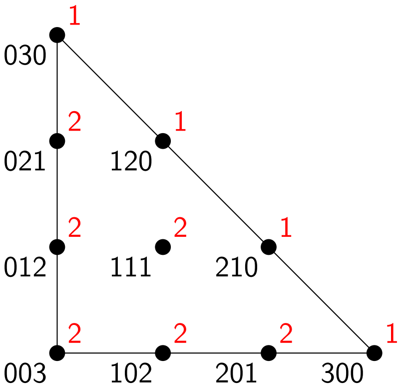

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Goal: Bounding the Minkowski volume

$$\mathrm{Vol}_{\mathbb{Z}^n}(P_1 + \cdots + P_n) = \sum_{\substack{(x_1,\ldots,x_n)\in Z^n_{\geqslant 0} \\ x_1+\cdots+x_n=n}} c_{(x_1,\ldots,x_n)} \,\mathrm{MV}(x_1,\ldots,x_n),$$

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- want to bound $\mathrm{Vol}_{\mathbb{Z}^n}(P_1 + \cdots + P_n)$ for fixed $\mathrm{MV}(P_1, \ldots, P_n)$
- idea: bound all summands $\mathrm{MV}(x_1, \ldots, x_n)$

Relations between mixed volumes

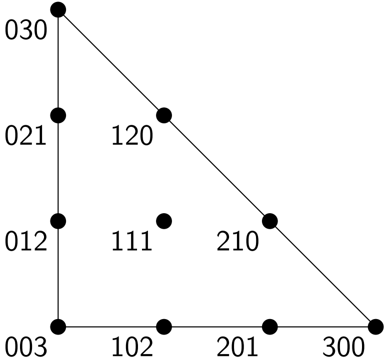
Theorem (Aleksandrov-Fenchel inequalities)

For $P_1, \ldots, P_n \subset \mathbb{R}^n$ lattice polytopes we have

$$\mathrm{MV}(P_1, \ldots, P_n)^2 \geqslant \mathrm{MV}(P_1, P_1, P_3, \ldots, P_n)\,\mathrm{MV}(P_2, P_2, P_3, \ldots, P_n),$$

or equivalently

$$\mathrm{MV}(x_1, \ldots, x_n)^2 \geqslant \mathrm{MV}(x_1 + 1, x_2 - 1, x_3, \ldots, x_n) \cdot \\ \cdot \mathrm{MV}(x_1 - 1, x_2 + 1, x_3, \ldots, x_n)$$



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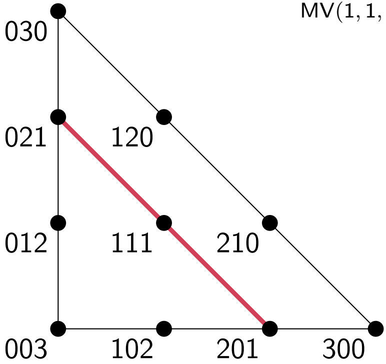

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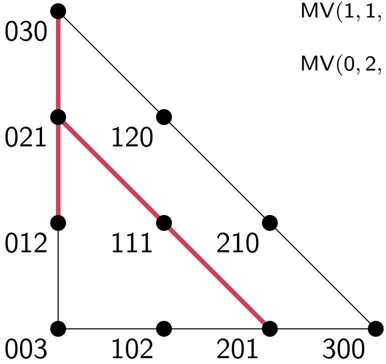

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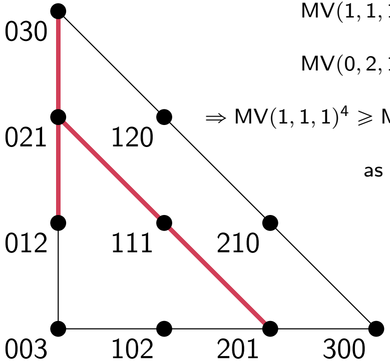

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$$\Rightarrow \mathrm{MV}(1,1,1)^4 \geqslant \mathrm{MV}(0,2,1)^2 \geqslant \mathrm{MV}(0,3,0) = \mathrm{Vol}_{\mathbb{Z}^n}(P_2)$$

as all mixed volumes $\geqslant 1$

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Concrete Bounds

Proposition (Schneider 14; Averkov, B, Soprunov 18+)

Denote $b_n(m) := \max\{\mathrm{Vol}_{\mathbb{Z}^n}(P_1 + \cdots + P_n)\colon \mathrm{MV}(P_1, \ldots, P_n) = m\}$ *(for full-dimensional polytopes). One has*

$$b_n(m) \in \begin{cases} \mathcal{O}(m^{3 \cdot 2^{\lfloor n/2 \rfloor}}) & \text{if } n \text{ is odd,} \\ \mathcal{O}(m^{2^{n/2}}) & \text{if } n \text{ is even.} \end{cases}$$

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- sharp w.r.t. Aleksandrov-Fenchel inequalities
- actually not sharp!

Theorem (Averkov, B, Soprunov, 18+)

Let $n \leqslant 6$. Then

$$b_n(m) \in \mathcal{O}(m^n).$$

Sharp bound, as $\mathrm{Vol}_{\mathbb{Z}^n}(\Delta_n + \cdots + \Delta_n + m\Delta_n) = (m+n-1)^n \in \mathcal{O}(m^n)$.

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- we can theoretically prove the bound for $n \leqslant 5$
- Question: find relations among the $\mathrm{MV}(x_1, \ldots, x_n)$ that show $b_n(m) \in \mathcal{O}(m^n)$ for general n!

Outline

State of classification for n-dimensional families of mixed volume m

What are all full-dimensional families $P_1, \ldots, P_n \subset \mathbb{R}^n$ with $\mathrm{MV}(P_1, \ldots, P_n) = m$?

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- $m = 1$, all P_i are translates of the same unimodular simplex [EG15]
- $n = 2$, $m \leqslant 4$ [EG16],
- proposed list for $n = 3$, $m = 2$,
- work in progress: $n = 3$, $m \in \{2, 3, 4, \dots\}$

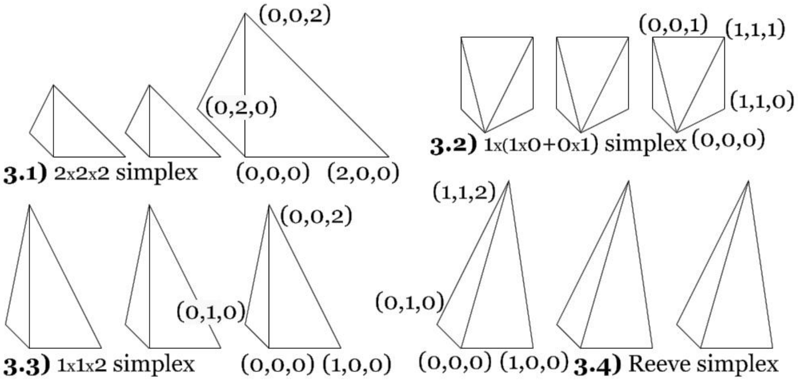

Figure: Proposed list of maximal families of mixed volume 2 by Esterov-Gusev

Conjecture (Esterov; Averkov, B, Soprunov)

For $m \ll n$, all full-dimensional families with $\mathrm{MV}(P_1, \dots, P_n) = m$ are contained in a family $\lambda_1 P, \dots, \lambda_n P$.

We need $m \ll n$ as e.g. for $n = 2$, $m = 2$ there are counterexamples.

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- $\mathrm{md}(P, \ldots, P) = \deg(P)$.

Connections to the mixed volume

Theorem (Esterov 15; Nill 17)

$$\mathrm{MV}(P_1, \ldots, P_n) = 1 \Leftrightarrow \mathrm{md}(P_1, \ldots, P_n) = 0.$$

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Theorem (Soprunov 07, Nill 17)

$$\mathrm{MV}(P_1, \ldots, P_n) - 1 \leqslant |\operatorname{int}_{\mathbb{Z}}(P_1 + \cdots + P_n)|,$$

with equality iff $\mathrm{md}(P_1, \ldots, P_n) = 1$.

Outline

Classification for mixed degree 1

Theorem (Batyrev, Nill 04)

$P \subset \mathbb{R}^n$ full-dimensional with $\deg(P) \leqslant 1$. Then either

- *P is the $(n-2)$-fold pyramid over $2\Delta_2$, or*
- *there is a lattice projection of P onto Δ_{n-1} (P is a Lawrence prism).*

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What about a mixed version of this?

Theorem (Balletti, B, 18+)

$P_1, \ldots, P_n \subset \mathbb{R}^n$ full-dimensional with $\mathrm{md}(P_1, \ldots, P_n) = 1$ and $n \geqslant 4$. Either

- *$P_1, \ldots, P_n$ is among finitely many exceptions, or*
- *$P_1, \ldots, P_n$ have common projection onto Δ_{n-1}.*

Classification for mixed degree 1

Theorem (Batyrev, Nill 04)

$P \subset \mathbb{R}^n$ *full-dimensional with* $\deg(P) \leqslant 1$. *Then either*

- *P is the $(n-2)$-fold pyramid over $2\Delta_2$, or*
- *there is a lattice projection of P onto Δ_{n-1} (P is a Lawrence prism).*

What about a mixed version of this?

Theorem (Balletti, B, 18+)

$P_1, \ldots, P_n \subset \mathbb{R}^n$ *full-dimensional with* $\mathrm{md}(P_1, \ldots, P_n) = 1$ *and* $n \geqslant 4$. *Either*

- *$P_1, \ldots, P_n$ is among finitely many exceptions, or*
- *$P_1, \ldots, P_n$ have common projection onto Δ_{n-1}.*

For $n = 3$ there exist infinitely many exceptions.

Outline

Questions about the mixed degree

- What are the exceptional families of mixed degree 1? Can they be described easily for n large enough?

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- What are the exceptional families of mixed degree 1? Can they be described easily for n large enough?
- Does the mixed volume yield an upper bound on the mixed degree (under some spanning conditions)?

Thank you!

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