Deformation cones for polytopes

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Polytopes

Definition

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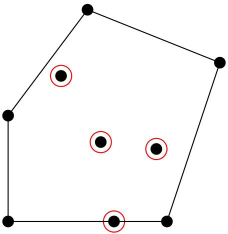

Alternative definition

Minkowski-Weyl Theorem

Every polytope P is the bounded intersection of finitely many halfspaces. More precisely, there exist $\mathbf{a}_1, \cdots, \mathbf{a}_n \in \mathbb{R}^d \setminus \{0\}$ and $b_1, \cdots, b_n \in \mathbb{R}$ such that

$$P = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}_1^t \mathbf{x} \leq b_1, \cdots, \mathbf{a}_n^t \mathbf{x} \leq b_n \right\}.$$

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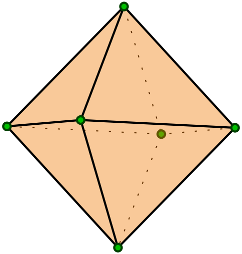
Example: Crosspolytope.

$$\Diamond_d = \mathrm{conv}(\pm \mathbf{e}_1, \cdots, \pm \mathbf{e}_d) = \left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{i=1}^{d} \pm x_i \leq 1 \right\}.$$



Interesting example: Permutohedron

$$\Pi_{d-1} = \mathrm{conv}\Big((\sigma(1), \cdots, \sigma(d)) : \sigma \text{ a permutattion} \Big),$$
$$= \left\{ \mathbf{x} \in \mathbb{R}^d : x_1 + \cdots + x_d = \binom{d+1}{2}, \sum_{i \in I} x_i \leq \binom{|I|+1}{2} \right\}.$$

This polytope has $d!$ vertices and $2^d - 2$ facets. Notice that its dimension is $d-1$.

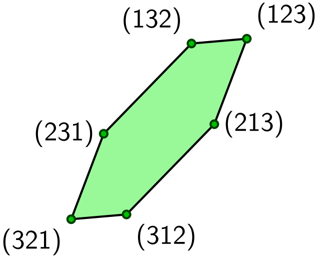
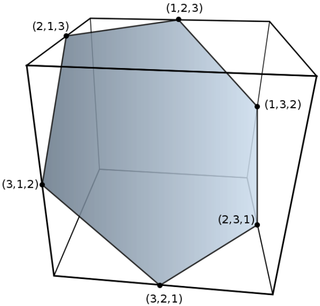


Figure: The 2-permutohedron.

2-permutohedron



$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 6 \\ & & x_2 & + & x_3 & \leq & 5 \\ x_1 & & & + & x_3 & \leq & 5 \\ x_1 & + & x_2 & & & \leq & 5 \\ x_1 & & & & & \leq & 3 \\ & & x_2 & & & \leq & 3 \\ & & & & x_3 & \leq & 3 \end{array}$$

Faces

For any $\mathbf{c} \in \mathbb{R}^d$ (which we will think of as a linear functional) define $P^{\mathbf{c}}$ as the points in P the farthest in direction $\mathbf{c}$.

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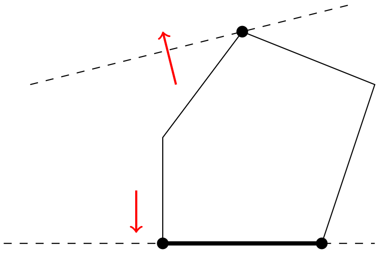

Figure: Two different faces.

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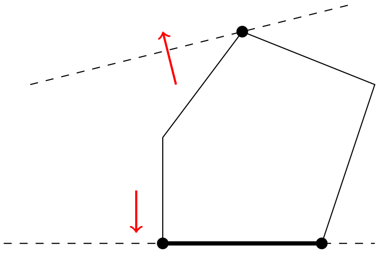

Figure: Two different faces.

A *facet* is a face of codimension 1.

Fact:

Each vertex is the intersection of the facets containing it.

Deformations

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We say Q is a deformation of P if we can obtain Q by moving the facets of P without overrunning vertices. That is, if a set of facets intersect in a vertex of P, the same set must intersect in a vertex of Q.

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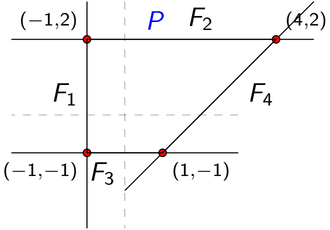
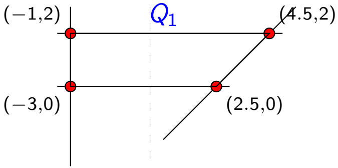
The main object of study is, given P, the set of **all** such Q.

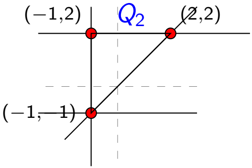
Example



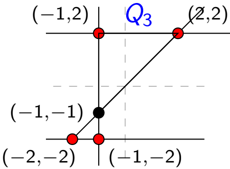
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$



$\mathbf{b}_1 = (3, 2, 0, 2.5)$



$\mathbf{b}_2 = (1, 2, 1, 0)$



$\mathbf{b}_3 = (1, 2, 2, 0)$

Example continued

What are all possible vectors **b** that give me a deformation?

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Answer for the previous example.

The set of all such vectors is given by the following two conditions

$$0 \leq b_2 + b_3, \qquad b_3 \leq b_1 + b_4.$$

A (non pointed) cone in $\mathbb{R}^4$.

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A (non pointed) cone in $\mathbb{R}^4$.

This is what we are looking for.

Sure, but why? Part I

1. Parameter spaces.

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When solving a problem (math or otherwise) it is always helpful to consider all possibilities simultaneously.

Don't fight in the North or the South. Fight every battle everywhere, always, in your mind. Everyone is your enemy, everyone is your friend. Every possible series of events is happening all at once. Live that way and nothing will surprise you. Everything that happens will be something that you've seen before. (Game of Thrones, Season 7, Episode 3)

Potentially useful for:

Are the following properties true over the whole cone?

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Sure, but why? Part II

2. Polytope Algebra.

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Q is a deformation of P iff exist λ, R such that $Q + R = \lambda P$. Another name is *weakly Minkowski summand*.

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Alternative parametrization.

In McMullen's theory deformations are parametrized by the edge lengths, (*balanced 1-weights*).

Sure, but why? Part III

3. Nef cones.

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MMP

In birrational geometry knowledge of the **facets** of the Nef cone is important.

Sure, but why? Part IV

4. Nice answers.

In the examples we are going to present, the resulting cones are interesting on their own. They all have the property that although we are considering exponentially many parameters, the inequalities have very small support (usually 4 or 5 terms).

Classical example: Permutohedron

Let's review the definition of permutohedron:

$$\begin{aligned}\Pi_{d-1} &= \mathrm{conv}\Big(\big(\sigma(1),\cdots,\sigma(d)\big) : \sigma \text{ a permutattion}\Big),\\ &= \left\{\mathbf{x} \in \mathbb{R}^d : x_1 + \cdots + x_d = \binom{d+1}{2}, \sum_{i \in I} x_i \leq \binom{|I|+1}{2}\right\}.\end{aligned}$$

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The takeaway from here is that the inequality description looks like

$$\{\mathbf{x} : \langle \mathbf{e}_I, \mathbf{x} \rangle = \sum_{i \in I} x_i \leq f(I) \qquad \forall I \subset [d]\}.$$

Remark/Question

To move facets mean to give a function on all subsets. Which functions give a deformation?

Submodular Theorem.

Theorem (Edmonds, Fujishige, Morton-Pachter-Shiu-Sturmfels-Wienand, C.-Liu)

A function $f : 2^{[d]} \longrightarrow \mathbb{R}$ gives a deformation of Π_{d-1} if and only if it is **submodular**. This means that

$$f(I) + f(J) \geq f(I \cap J) + f(I \cup J).$$

for any two subsets.

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- Not all of them are necessary. The essential ones (facet-defining) are of the form $f(I \cup \{a\}) + f(I \cup \{b\}) \geq f(I) + f(I \cap \{a, b\})$, where $a, b \notin I$.

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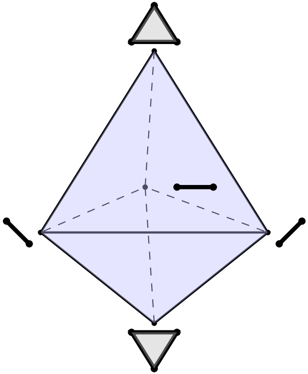
- Matroid polytopes, and more general, polymatroids are deformations of permutohedra.

Submodular cone

After quotienting by the lineality space, the deformation cone of Π_2 is the 4-dim cone over the following picture:



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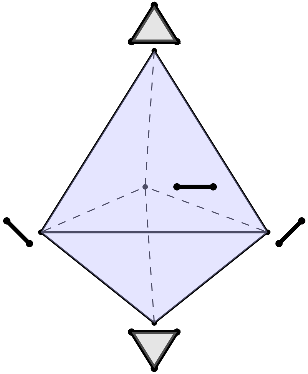

We know the facets. The rays are unknown and quite an interesting problem. Loopless Matroid polytopes are extremal, but there are extremal rays not coming from matroids.

Extending to Coxeter Arrangements

One important aspect of the permutohedron is that it has lots of symmetry. Computation of the Deformation cone is certainly simpler when there is symmetry involved. In a precise way the symmetry present in Π_{d-1} can be defined and classified.

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\{\text{Braid fan}\} &\longrightarrow \quad \{\text{Coxeter arrangement}\} \\
\{\text{Symmetric group } S_d\} &\longrightarrow \quad \{\text{Weyl group } W\} \\
\{\text{Permutohedron } \Pi_{d-1}\} &\longrightarrow \quad \{W\text{-permutohedron (sum of all positive root} \\
\{\text{facets indexed by subsets}\} &\longrightarrow \quad \{\text{facets indexed by weights}\}
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This particular generalization comes up in the study of *Coxeter Matroids*.

W-submodular

By using carefully the same methods as in the permutohedron, we get that a that:

Theorem (Ardila-C.-Postnikov)

The submodular cone is given by

$$\sum_{j \neq i} \left(-2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \right) f(w_j) \leq f(w_i) + f(s_i w_i),$$

and all other inequalities obtained by applying W to the equation. Here w_k's are the fundamental weights, s_i is the reflection that fixes all fundamental weights except w_i.

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Remarks:

- In type A this recovers submodular theorem.
- Most coefficients are zero. The only nonzero are the neighbours in the Dynkin diagram.

Another generalization

In recent work with Liu, we defined a nested permutohedron. Informally, we replace each vertex of the permutohedron by a smaller dimension permutohedron.

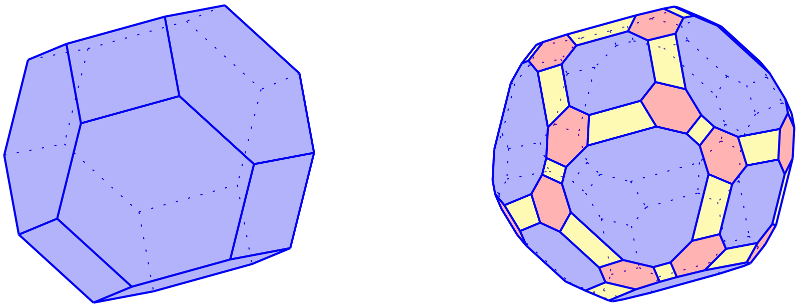

Figure: Π_3 and $\Pi_3^2(4, 1)$

The actual definition

Remark

We actually define the new polytope through is normal fan. The braid fan classifies points according to the relative order of the entries. The nested braid fan classifies points according to the relative order of the entries **and** the relative order of the first differences.

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The 'doubly' nested braid fan is **not** the next barycentric subdivision.

One key point

Facets are indexed by ordered set partitions, a combinatorial object.

Posets

In the permutohedron case, facets where indexed by sets.

Remark

In this case, facets form a poset. In general there is no natural poset structure on the set of facets.

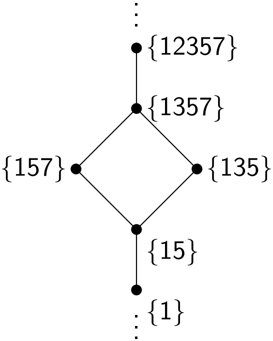
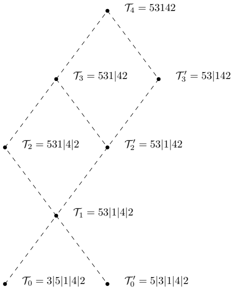


Figure: Diamond giving the inequality $f(\{135\}) + f(\{157\}) \geq f(\{1357\}) + f(\{15\})$

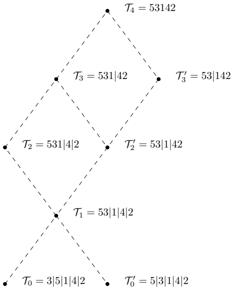
Deformation for nested permutohedron

In this case, facets are indexed by ordered partitions and they also form a poset. The deformation cone have two types of inequalities.



1. For each diamond, same "submodular" pattern.
2. For elements in the bottom, we have something called the ⅄ condition.

Examples of inequalities



We have inequalities

$$f(531|4|2) + f(53|1|42) \geq f(531|42) + f(53|1|4|2) \quad \text{Diamond}$$
$$f(3|5|4|2) + f(5|3|1|42) + f(53142) \geq 2f(53|1|42) + f(53|142) \quad \text{Ren}$$

Examples of inequalities

One more to see the general ⅄ pattern:

$$f(1|9|\mathbf{2}|\mathbf{8}|3|6|7|4|5) + f(1|9|\mathbf{8}|\mathbf{2}|3|6|7|4|5) + f(192836745) \geq 2f(1|9|\mathbf{28}|3|6|7|4) + f(19|\mathbf{28}|36745)$$

Theorem(C.-Liu)

The collection of all diamonds and ⅄ inequalities give the deformation cone of the nested permutohedron.

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Potential uses:

Are the following properties true over the whole cone?

1. Normality.
2. Ehrhart Positivity.
3. Unimodality of δ-vector.
4. Quadratic generation of toric ideal.

The End.

¡Gracias!