Classification of Minimal Polygons with Specified Singularity Content

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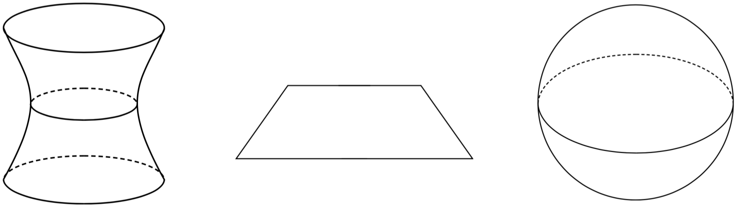
Fano Varieties

A projective manifold, or a variety X over $\mathbb{C}$ comes equipped with a notion of curvature. We use this to place X into one of three classes:

General type	*Calabi – Yau*	*Fano*



Negative Curvature	Flat	Positive Curvature
Infinitely Many	Unknown	Finitely Many

It is natural to ask for a classification of smooth Fano varieties in dimension n.

Fano Varieties

Smooth varieties (manifolds) are classified up to dimension 3.

Dimension	Number of Smooth Fano Varieties	Examples
1	1	$\mathbb{P}^1$
2	10	$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and blow-up of $\mathbb{P}^2$ in ≤ 8 points
3	105	$\mathbb{P}^3$, etc.

The 2-dimensional classification is due to del Pezzo in 1880s and the 3-dimensional classification due to Mori–Mukai in 1980s. We do not know much in dimension greater or equal than 4.

Fano Polytopes

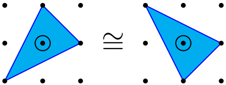
Let N be a lattice. A *polytope* P in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ is a set of the form

$$P = \left\{ \sum_{u \in S} \lambda_u u : \lambda_u > 0 \text{ and } \sum_{u \in S} \lambda_u = 1 \right\},$$

where $S \subset N_{\mathbb{R}}$ is a finite set of points. A *Fano polytope* is a full-dimensional convex polytope such that

- the vertices $\mathcal{V}(P) \in N$ are all primitive.
- the origin lies in the strict interior of P.

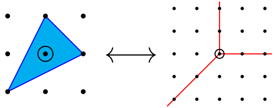
When N is a rank-two lattice P is known as a *Fano polygon*. We consider polytopes up to $GL(N)$-equivalence.



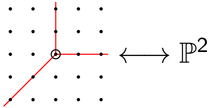
$\cong$ via the change of basis $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ on N.

Fano Polytopes

The span of each face E of a Fano polygon P, by which we mean $\mathbb{R}_{\geq 0}E$, defines a cone. We obtain a fan in $N_{\mathbb{R}}$ corresponding to P.

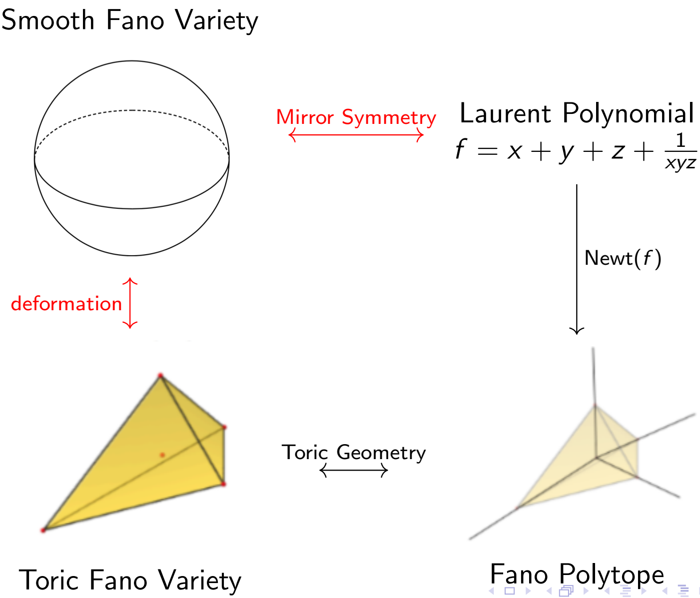


This determines a *toric* del Pezzo surface X_P.



Many properties of X_P have combinatorial analogues in the Fano polygon P; examples include the singularities and the anticanonical degree $(-K_{X_P})^2$.

Mirror Symmetry



Mirror Symmetry

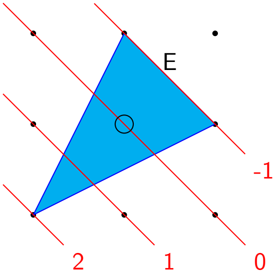
Given a Fano manifold X, mirror symmetry tells us (conjecturally) how to associate a Laurent polynomial f which is said to be *mirror dual* to X.

- Choice of mirror dual is not necessarily unique.
- Can transform f via a *mutation* (a special birational transformation $\phi : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$) to obtain another Laurent polynomial g
- g also mirror dual to X.
- This notion of a mutation is captured when we move to the Fano polytopes $\mathrm{Newt}(f)$ and $\mathrm{Newt}(g)$.

Mutation of Polygons

Let $P \subset N_{\mathbb{R}}$ be a polygon, and E be an edge of P. Consider the primitive inward pointing normal $\omega_E \in M = \mathrm{Hom}(N, \mathbb{Z})$ of this edge. This vector can be thought of as a grading function on the polygon P. For $h \in \mathbb{Z}$, define

$$\omega_h(P) = \mathrm{conv}\{v \in N \cap P : \omega_E(v) = h\}.$$



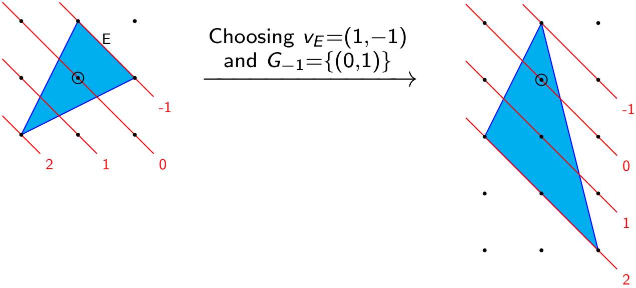
Mutation of Polygons

Choose v_E to be a primitive vector of the lattice N such that $\omega_E(v_E) = 0$. Set $F = \mathrm{conv}\{\mathbf{0}, v_E\}$. For all $h < 0$, suppose that there exists $G_h \subset N_{\mathbb{R}}$ such that

$$\{v \in \mathcal{V}(P) : \omega_E(v) = h\} \subseteq G_h + |h|F \subseteq \omega_h(P).$$

Then we define the *mutation* of P given by ω_E, F and G_h to be

$$\mathrm{mut}_{(\omega_E,F)}(P) = \mathrm{conv}\Big(\bigcup_{h<0} G_h \cup \bigcup_{h\geq 0}(\omega_h(P) + hF)\Big) \subset N_{\mathbb{R}}$$



Mutation of Polygons

Lemma

Let E be an edge of a Fano polygon P with primitive inner normal vector $\omega_E \in M$. Then P admits a mutation with respect to ω if and only if $|E \cap N| - 1 \geq |\omega(E)|$.

We can use mutations to define an equivalence relation on the set of Fano polygons.

Cyclic Quotient Singularities

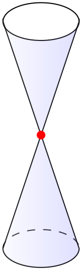
A *quotient singularity* $\frac{1}{R}(a, b)$ is given by the action of μ_R on $\mathbb{C}^2$ by $(x, y) \mapsto (\epsilon^a x, \epsilon^b y)$ where ϵ is an R^{th} root of unity, and considering $Z = \operatorname{Spec}(\mathbb{C}[x, y]^{\mu_R})$. The germ of the origin is the singularity.

For example consider a $\frac{1}{2}(1, 1)$ singularity. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $\epsilon = -1$. We consider the action of G on $\mathbb{C}^2$ described by

$$-1 \cdot (x, y) = (-x, -y).$$

We have

$$\begin{aligned} Z &= \operatorname{Spec}\left(\mathbb{C}[x, y]^G\right) \\ &= \operatorname{Spec}\left(\mathbb{C}[x^2, xy, y^2]\right) \\ &= \operatorname{Spec}\left(\mathbb{C}[u, v, w]/(uw - v^2)\right) \\ &= \mathbb{V}(uw - v^2) \subset \mathbb{C}^3. \end{aligned}$$



Cyclic Quotient Singularities

A quotient singularity $\frac{1}{R}(a, b)$ is cyclic if:

$$\gcd(R, a) = \gcd(R, b) = 1.$$

Set:

$$k = \gcd(a + b, R).$$

So:

$$a + b = kc \text{ and } R = kr.$$

We can write the cyclic quotient singularity as

$$\frac{1}{kr}(1, kc - 1).$$

Cyclic Quotient Singularities

We have two types of singularities here:

- A cyclic quotient singularity $\frac{1}{kr}(1, kc-1)$ is a *T-singularity* if $r \mid k$.
  - A T-singularity admits a qG-smoothing. (Kollar–Shepherd-Barron)
- A cyclic quotient singularity $\frac{1}{kr}(1, kc-1)$ is an *R-singularity* if $k < r$.
  - An R-singularity is rigid under qG-deformation. (Kollar–Shepherd-Barron)

Singularity Content

Consider an arbitrary cyclic quotient singularity $\sigma = \frac{1}{kr}(1, kc-1)$. By the Euclidean Algorithm there exists unique non-negative integers n and k_0 such that:

$$k = nr + k_0.$$

If $k_0 > 0$, then σ is qG-deformation equivalent to a $\frac{1}{k_0 r}(1, k_0 c - 1)$ cyclic quotient singularity. The *residue* of σ is given by:

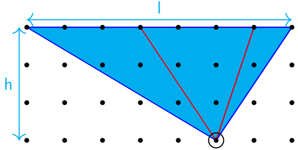
$$\mathrm{res}(\sigma) = \begin{cases} \varnothing & \text{, if } k_0 = 0 \\ \frac{1}{k_0 r}(1, k_0 c - 1) & \text{, otherwise.} \end{cases}$$

The *singularity content* of σ is given by the pair:

$$\mathrm{SC}(\sigma) = \big(n, \mathrm{res}(\sigma)\big).$$

Singularity Content

Consider a cone C corresponding to an edge E of a polygon:



By the Euclidean algorithm:

$$l = hn + r.$$

We divide C into separate sub-cones $C_0, \cdots, C_n$ where $C_1, \cdots, C_n$ (known as T-cones) have lattice length h, and C_0 has lattice length r and is known as an R-cone.

Singularity Content

Each cone corresponds to a cyclic quotient singularity of the corresponding toric variety. This allows us to define the singularity content of an edge E.

Let $P \subset N_{\mathbb{R}}$ be a polygon.

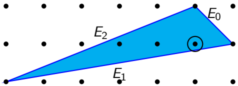
- Label the edges of P in clockwise order $E_1, \cdots E_k$.
- Each edge E_i corresponds to a cyclic quotient singularity σ_i corresponding to this cone.
- Let $\mathrm{SC}(E_i) = \big(n_i, \mathrm{res}(\sigma_i)\big)$.
- We define the *singularity content* of P to be:

$$\mathrm{SC}(P) = \Big(\sum_{i=1}^{k} n_i, \mathcal{B}\Big),$$

where $\mathcal{B} = \{\mathrm{res}(\sigma_1), \cdots, \mathrm{res}(\sigma_k)\}$.

Singularity Content

Consider the following polygon P.



- E_0 and E_1 both give T-cones.
- E_2 gives an R-cone representing a $\frac{1}{5}(1,1)$ singularity.

So P has singularity content $\left(2, \{\frac{1}{5}(1,1)\}\right)$

Singularity content is an invariant under mutation!

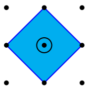
Classification of Fano Polygons

Conjecture A: There exists a bijective correspondence between the set of mutation-equivalence classes of Fano polygons and the set of qG-deformation equivalence classes of locally qG-rigid TG del Pezzo surfaces with cyclic quotient singularities.

Recent results from Corti, Heuberger, Kasprzyk, Nill, Prince certainly support this conjecture.

Classification of Fano Polygons

- There are precisely 10 mutation-equivalence classes of Fano polygons with singularity content $(n, \emptyset)$. They are in bijective correspondence with the 10 families of smooth del Pezzo surfaces.
- There are precisely 26 qG-deformation families of del Pezzo surfaces with $m \geq 1$ singular points of type $\frac{1}{3}(1,1)$ admitting a toric degeneration. They are in bijective correspondence with 26 mutation-equivalence classes of Fano polygons with singularity content $(n, \{m \times \frac{1}{3}(1,1)\}), m \geq 1$.



$$\longleftrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

Classification of Fano Polygons

Kutas and C. have designed an efficient algorithm to build on the work of Kasprzyk, Nill and Prince.

Input: Singularity Content $(n, \mathcal{B})$.
Output: Representative of every mutation-equivalence class of Fano polygons with singularity content $(n, \mathcal{B})$.

Assuming Conjecture A holds, this is equivalent to a classification of locally qG-rigid del Pezzo surfaces admitting a toric degeneration.

Classification of Fano Polygons

As a corollary to this algorithm we have the following classifications:

- There are precisely 14 mutation-equivalence classes of Fano polygons with singularity content $\left(n, \{m_1 \times \frac{1}{3}(1,1), m_2 \times \frac{1}{6}(1,1)\}\right)$ with $m_1 \geq 0, m_2 > 0$.
- There are precisely 12 mutation-equivalence classes of Fano polygons with singularity content $\left(n, \{m \times \frac{1}{5}(1,1)\}\right)$ with $m > 0$.

Classification of Fano Polytopes

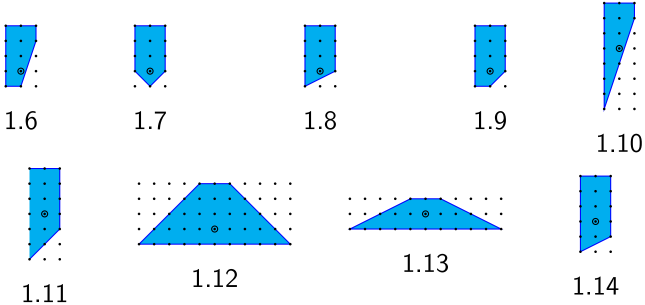


Figure 1: Minimal Representatives of Mutation-Equivalence Classes of Fano Polygons with Singularity Content $\left(n, \{m_1 \times \frac{1}{3}(1,1), m_2 \times \frac{1}{6}(1,1)\}\right)$ where $m_1 \geq 0$, $m_2 > 0$.