Covering minima of lattice polytopes

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joint work with Francisco Santos and Matthias Schymura

Volume of centrally symmetric convex bodies

Theorem (Minkowski, 1889)

Let Λ be a lattice in $\mathbb{R}^n$, and $K \subset \mathbb{R}^n$ a convex body which is centrally symmetric w.r.t. the origin and contains no other lattice points, then

$$vol(K) \leq 2^n \det(\Lambda)$$

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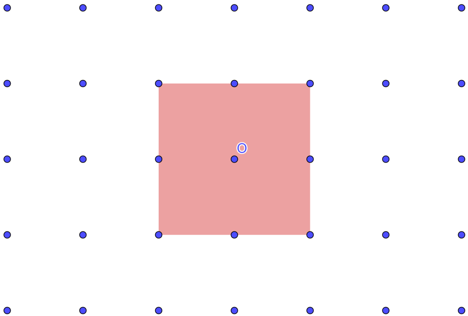

$\Lambda = \mathbb{Z}^2$

Non-symmetric bodies

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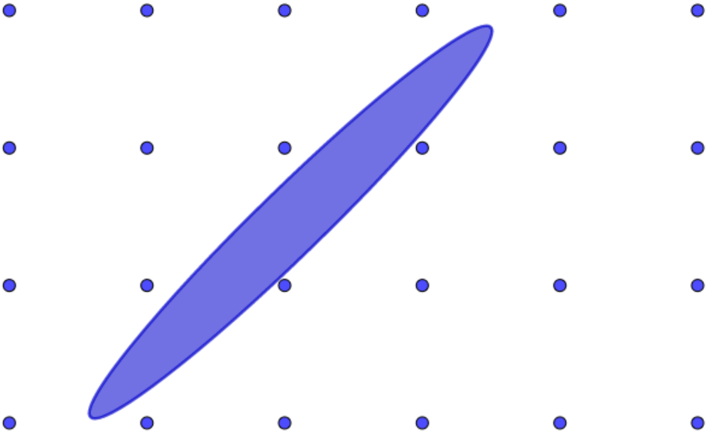
Pancakes can have arbitrarily large volume!

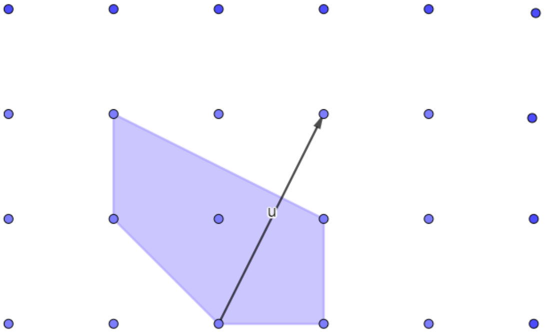


Lattice width

For a vector $u \in \Lambda^*$, the **width** of a convex body K w.r.t u is

$$\omega(K,u) = \max_{v_1,v_2 \in K} (< v_1, u > - < v_2, u >),$$

which is $|\{\text{lattice hyperplanes with normal } u \text{ which meet } K\}| - 1$ if K is a lattice polytope.



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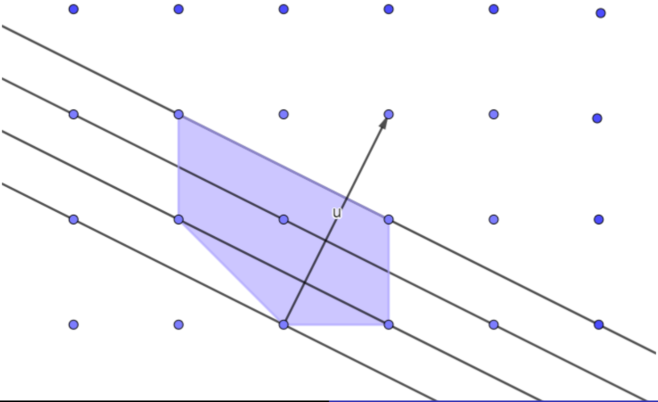

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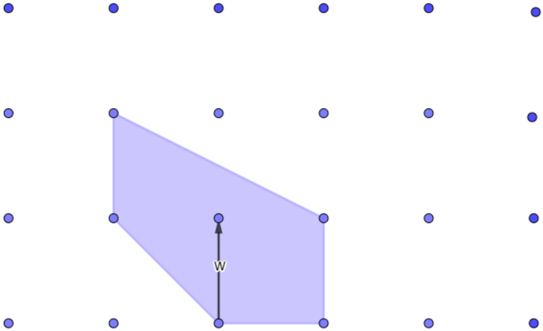

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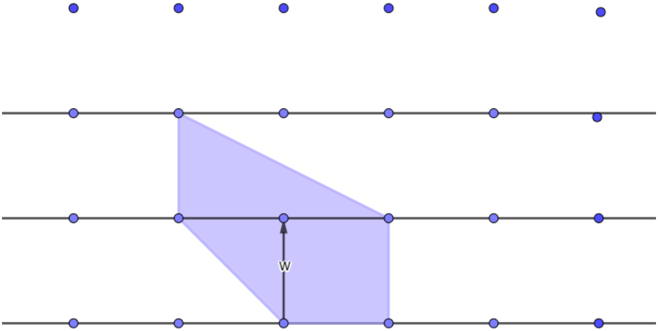

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Kannan-Lovasz, 1988: $f(n) \in \mathcal{O}(n^2)$.

Covering minima

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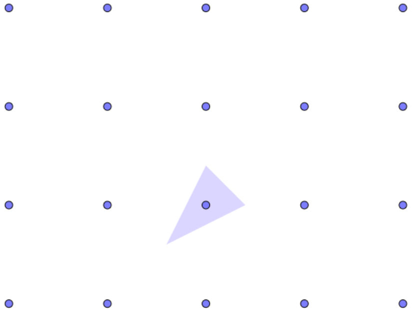
The i-**th covering minimum** of a convex body $K \subseteq \mathbb{R}^n$ w.r.t. Λ is

$$\mu_i(K, \Lambda) = \min\{\mu \geq 0 : \mu K + \Lambda \cap L \neq \emptyset \text{ for all } (n-i)\text{-dim'l affine subspaces } L\}$$



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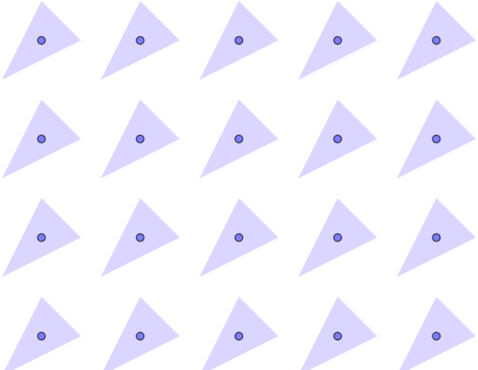

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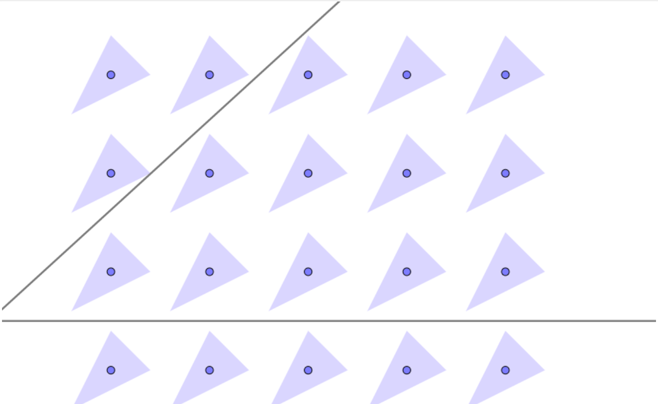

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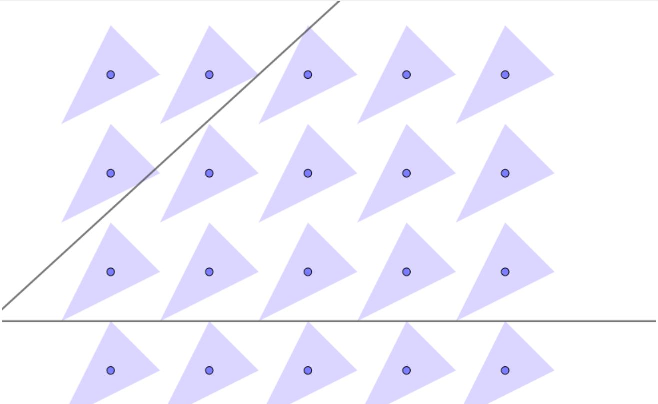

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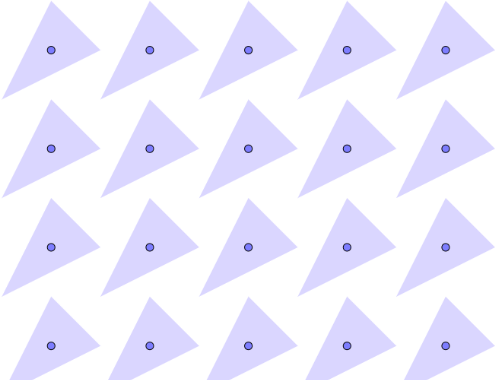

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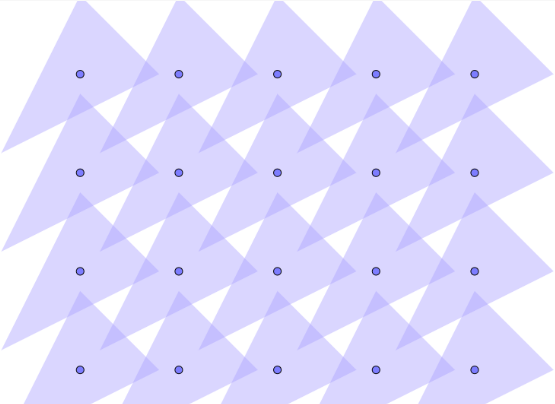

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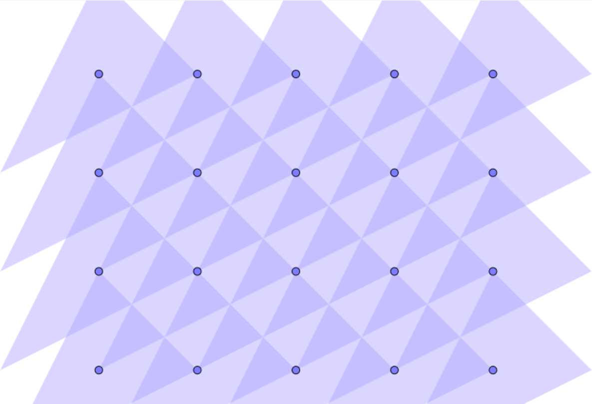

The standard simplex

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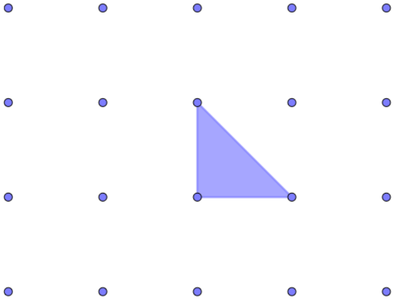
$$\mu_i = ?$$



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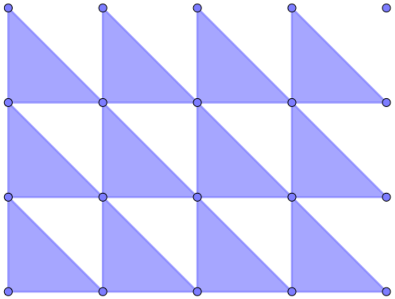

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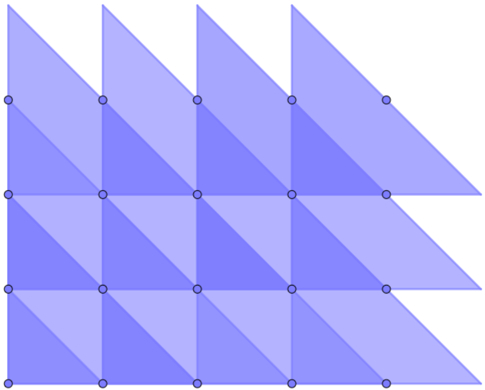

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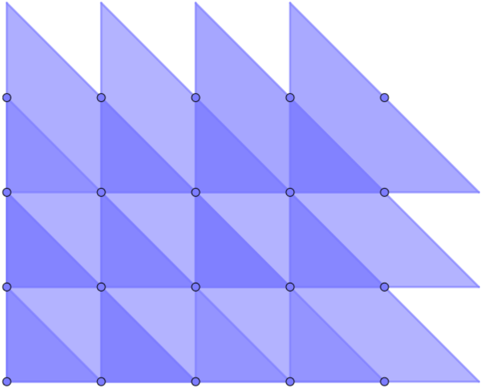

First observations on covering minima

For a convex body $K \subseteq \mathbb{R}^n$ and a lattice Λ,

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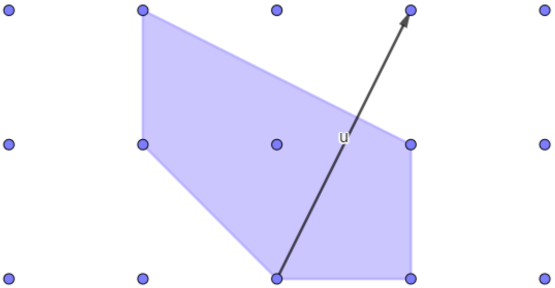

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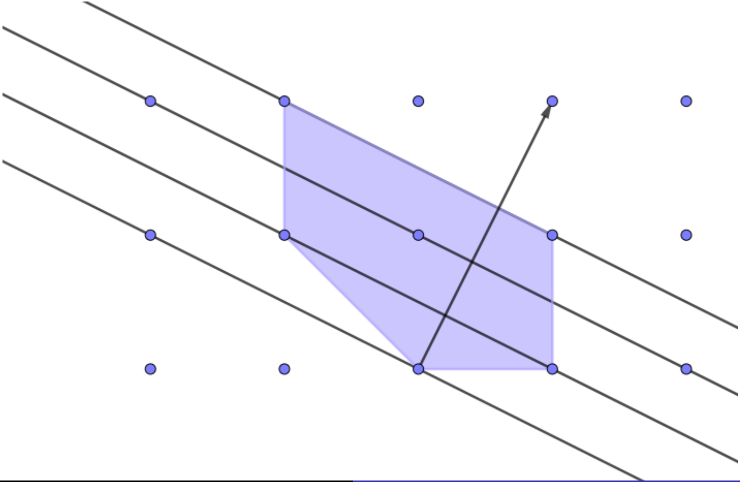

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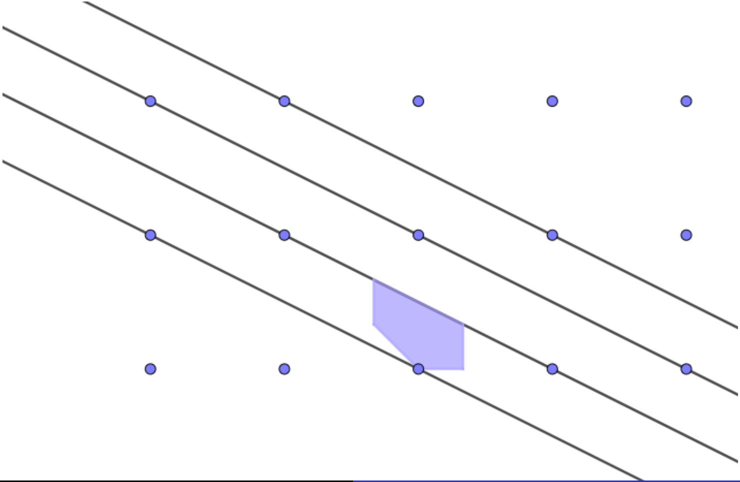

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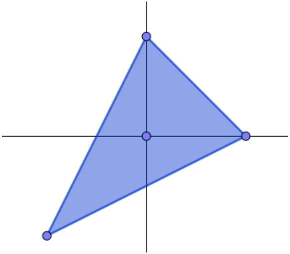
- Used the interplay of the covering minima to prove a result about width (flatness theorem);
- They did much more! These covering minima are (in some sense) analogues of the **successive minima** of Minkowski, and their goal was to prove bounds for the volume with similar flavor to Minkowski's theorems.

Our conjectures and their equivalences

What happens for the simplex $S_n^1 = \mathrm{conv}\{e_1, \dots, e_n, -\mathbf{1}\}$?



Conjecture (Schymura-Gonzales)

The i-th covering minimum of S_n^1 satisfies

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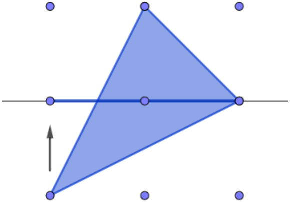

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$$\mu_{n-1}(S_n^1, \mathbb{Z}^n) \geq \mu_{n-1}(\pi(S_n^1), \pi(\mathbb{Z}^n)) = \mu_{n-1}(S_{n-1}^1, \mathbb{Z}^{n-1}) = \tfrac{n-1}{2}$$

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We could show that the previous conjecture is equivalent to the following conjecture concerning the covering radius:

Conjecture

For any P lattice polytope with at least 1 interior lattice point, we have

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We could show that the previous conjecture is equivalent to case $k = 1$ of the following conjecture concerning the covering radius:

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For any P lattice polytope with at least k interior lattice points, $k \in \mathbb{Z}_{\geq 0}$, we have

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where $S_n^k = \mathrm{conv}\{e_1, \ldots, e_n, -k\mathbf{1}\}$. So what is $\mu_n(S_n^k, \mathbb{Z}^n)$?

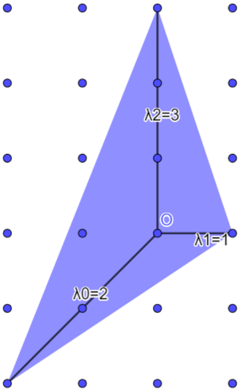
Conjectured maximum

Given $\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{Z}^{n+1}_{>0}$, we define the simplex

$$S_\lambda = \mathrm{conv}(-\lambda_0 \mathbf{1}, \lambda_1 e_1, \dots, \lambda_n e_n).$$

Theorem (C-Santos-Schymura '18+)

$$\mu_n(S_{\lambda_0,\dots,\lambda_n}) = \frac{\sum_{i,j\in[n],i<j} \left(\prod_{k\neq i,k\neq j} \lambda_k \right)}{\sum_{i\in[n]} \prod_{k\neq i} \lambda_k}$$



Covering radius vs volume and surface area

This leads us to a (final!) conjecture relating the covering radius to a quotient of "lattice surface area" and lattice volume:

Conjecture

Let $P = \text{conv}(v_0, \ldots, v_n)$ be a lattice simplex, and $p \in int(P)$. Denote by π_i the projection along $\overrightarrow{pv_i}$. Then:

$$\mu_n(S, \mathbb{Z}^n) \leq \frac{\sum_{i=0}^{n} \text{Vol}(\pi_i(S), \pi_i(\mathbb{Z}^n))}{2\,\text{Vol}(S, \mathbb{Z}^n)}.$$

Covering radius vs volume and surface area

For us this is exciting because it would be a discrete analogue of the following theorem of Wills:

Theorem (Wills, 1968)

For every proper convex body $K \subseteq \mathbb{R}^n$

$$\mu_n(K, \mathbb{Z}^n) \leq \frac{\operatorname{surf}(K)}{2\operatorname{vol}(K)},$$

where $\operatorname{vol}(K)$ *and* $\operatorname{surf}(K)$ *are the Euclidean volume and surface area of K.*