Cubical Dehn-Sommerville equations

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What are the classical Dehn-Sommerville Equations?

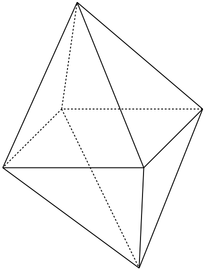
Definition

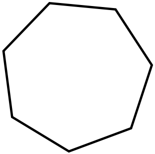
A polytope is *simplicial* if all of its faces are simplices.

Theorem (Dehn, Sommerville, 1905-1927)

Let P be a simplicial d-dimensional polytope with face-vector $(f_{-1}, \ldots, f_d)$. Then for $0 \leq j \leq d$:

$$f_{j-1} = \sum_{k=j}^{d} (-1)^{d-k} \binom{k}{j} f_{k-1}$$





How else can we formulate this?

- Recall from Katharina's problem sheet that we can define an $h - vector$ for simplicial polytopes:

$$\sum_{i=0}^{d} f_{i-1} z^i (1-z)^{d-i} = \sum_{i=0}^{d} h_k z^i.$$

- The Dehn-Sommerville equations can be stated as a symmetry in the h-vector:

$$h_i = h_{d-i}$$

Dehn-Sommerville Equations for cubical polytopes

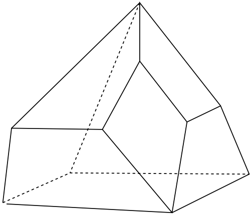
Definition

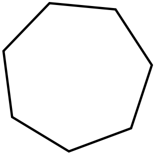
A polytope is *cubical* if all of its faces are combinatorially equivalent to cubes.

Theorem

Let P be a cubical d-dimensional polytope with face-vector $(f_{-1}, \ldots, f_d)$. Then for $0 \leq j \leq d$, we have:

$$f_{j-1} = \sum_{k=j}^{d} (-1)^{d-k} 2^{k-j} \binom{k-1}{j-1} f_{k-1}$$





Questions!!!

- There are generalizations of Dehn-Sommerville equations for simplicial polytopes! Can these generalizations be extended as far in the cubical case?
- There are many proofs of Dehn-Sommerville. Do these proof techniques work well in the cubical case?
- Can we express the cubical Dehn-Sommerville as a symmetry relation of some cubical h-vector?

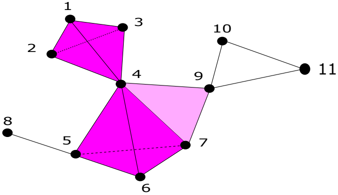
How far can one generalize Dehn-Sommerville?

Definition

An *(abstract) simplicial complex* is a nonempty collection Γ of subsets of a finite set V such that:

- if $\sigma \in \Gamma$ and $\sigma' \subset \sigma$, then $\sigma' \in \Gamma$.

We call the σ *faces* of Γ.



Maximal Faces

{1,2,3,4}
{4,5,6,7}
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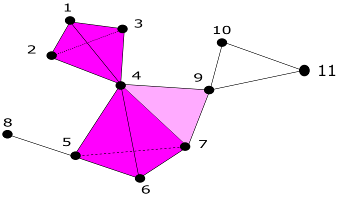

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Note: Every (abstract) simplical complex Γ can be realized as the face poset of a complex of unimodular simplices!

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Theorem

Let Γ *be a* d*-dimensional Eularian simplicial complex. Then for* $0 \leq i \leq d+1$,

$$h_i = h_{d-i+1}$$

which is equivalent to

$$f_{j-1} = \sum_{k=j}^{d+1} (-1)^{d+1-k} \binom{k}{j} f_{k-1}$$

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Proofs outlined in Beck-Sanyal's new book:

- uses the idea of self-reciprocol complexes.
- uses the idea of chain partition functions

Generalized form for cubical complexes?

Definition

A cubical complex Γ is a nonempty collection of subsets on a finite set V closed under intersection such that:

- $\{v\} \in V$ for all $v \in V$
- For every face $\sigma \in \Gamma$ the interval $[\emptyset, \sigma]$ is isomorphic to the lattice of faces of a cube.

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Proof: uses flags, defines a long and a short cubical h-vector!!!

What is a self-reciprocal complex?

Let $\mathcal{K}$ be a complex of lattice polytopes in $\mathbb{R}^d$. We can extend the definition of the ehrhart polynomial:

$$ehr_{\mathcal{K}} = |n\mathcal{K} \cap \mathbb{Z}^d|$$

This always agrees with a polynomial of degree $\dim(\mathcal{K})$.

Definition

$\mathcal{K}$ is *self-reciprocal* if for all $n > 0$:

$$(-1)^{dim(\mathcal{K})}\text{ehr}_{\mathcal{K}}(-n) = \text{ehr}_{\mathcal{K}}(n)$$

In terms of h^*-vector

Recall from Katharina's lectures:

Definition

$$Ehr_{\mathcal{K}}(z) := 1 + \sum_{n \geq 1} ehr_{\mathcal{K}}(n) z^n = \frac{h_0^* + h_1^* z + \cdots + h_{d+1}^* z^{d+1}}{(1-z)^{d+1}}$$

We call $(h_0^*, \ldots h_{d+1}^*)$ the h^*-*vector* of $\mathcal{K}$.

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Theorem

Let $\chi(\mathcal{K}) = 1 - (-1)^{d+1}$. Then $\mathcal{K}$ is self-reciprocal if and only if:

$$h_{d+1-i}^*(\mathcal{K}) = h_i^*(\mathcal{K})$$

What are examples of self-reciprocal complexes?

Proposition

The boundary complex (all proper faces) of any lattice polytope is self-reciprocal.

Proof uses Ehrhart-Mcdonald reciprocity.

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Again, use Ehrhart-Mcdonald reciprocity.
In all these cases, $\chi(\mathcal{K}) = 1 - (-1)^{d+1}$ so the h^* vector is symmetric!

How do we use this to prove generalized Dehn-Somerville?

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- Generalized Dehn-Sommerville!

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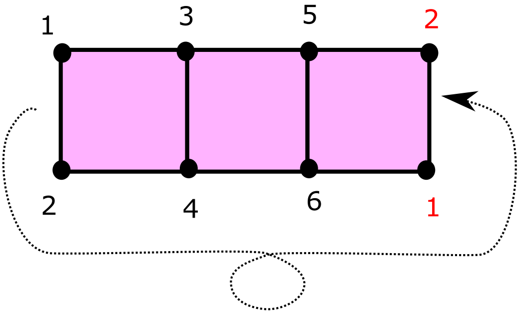

What are chain partitions of a poset?

Let Π be a finite poset with $\hat{0}, \hat{1}$. Let $\phi : \Pi \to \mathbb{Z}_+$ be an order-preserving map.

Definition

A $(\Pi, \phi)-$*chain partition* of n is of the form:

$$n = \phi(c_1) + \cdots + \phi(c_m)$$

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Theorem

Let Π be Eulerian of rank $d+1$, and let ϕ be ranked. Then,

$$(-1)^d cp_{\Pi,\phi}(-n) = cp_{\Pi,\phi}(n - \phi_1 - \cdots - \phi_d)$$

Chain partitions of simplicial complexes?

Let Γ be a simplicial complex on V.
Consider: multisubset represented as $\mathbf{a} \in \mathbb{Z}_{\geq 0}^V$, where a_v is the multiplicity of v.

$$\text{supp}(\mathbf{a}) := \{v \in V | a_v > 0\}$$

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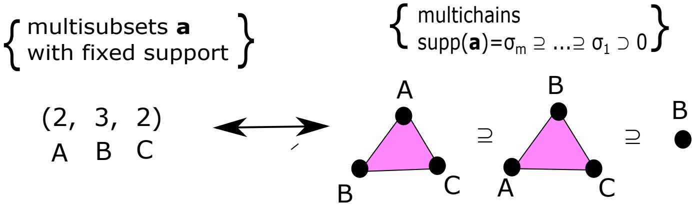

From chain partitions to Dehn-Sommerville

Theorem (Beck, Sanyal)

Let Γ *be a simplicial complex of dimension* $d-1$ *with* $\phi(\sigma) = |\sigma|$*.*
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- Dehn Sommerville!!

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I hope so!

Thank you all!!! :) :)