# The complete classification of lattice empty $4$-simplices

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# Empty lattice $d$-simplices

## Definition

A *lattice polytope* $P \subset \mathbb{R}^d$ is a polytope with integer vertices. It is:

- *hollow* if it has no integer points in its interior.
- *empty (lattice-free)* if it has no integer points other than its vertices.

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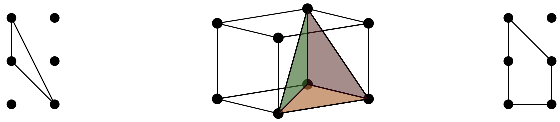

Empty $2$ and $3$-simplices and hollow $2$-polytope.

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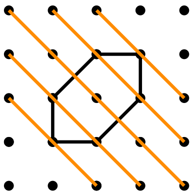
- The <u>*width*</u> of $P \subset \mathbb{R}^d$ with respect to a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ equals the difference $\max_{x \in P} f(x) - \min_{x \in P} f(x)$.



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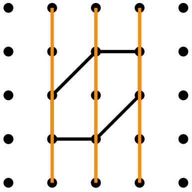
  We call *(lattice) width* of $P$ the minimum width of $P$ with respect to integer functionals.



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- The only empty $1$-simplex is the unit segment.
- The only empty $2$-simplex is the unimodular triangle ($\simeq$ Pick's Theorem).
- There are infinitely many of width one (Ex: Reeve polyhedra). Empty lattice $3$-simplices are completely classified:

**Theorem (White 1964)**

*Every empty tetrahedron of determinant $q$ is equivalent to*

$$T(p, q) := \mathrm{conv}\{(0,0,0), (1,0,0), (0,0,1), (p,q,1)\}$$

*for some $p \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Moreover, $T(p, q) \cong_{\mathbb{Z}} T(p', q)$ if and only if $p' = \pm p^{\pm 1} \pmod{q}$.*

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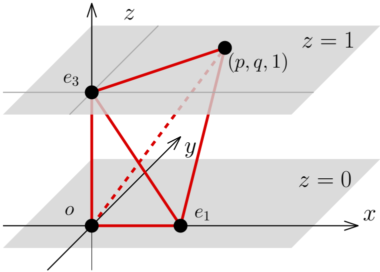
In particular, they all have width 1, i.e., they are between two parallel lattice hyperplanes.




In this picture, they have width $1$ with respect to the functional $f(x, y, z) = z$.

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**Proposition (Blanco-Haase-Hofmann-Santos, 2016)**

1. For each $d$, there is a $w^\infty(d)$ such that for every $n \in \mathbb{N}$ all but finitely many $d$-polytopes with $n$ lattice points have width $\leq w^\infty(d)$.
2. $w^\infty(4) = 2$.

# What do we know about empty lattice $4$-simplices?

**Theorem (Haase-Ziegler, 2000)**

*Among the $4$-dimensional empty simplices with width greater than two and determinant $D \leq 1000$,*

1. *All simplices of width 3 have determinant $D \leq 179$, with a (unique) smallest example, of determinant $D = 41$, and a (unique) example of determinant $D = 179$.*
2. *There is a unique class of width 4, with determinant $D = 101$,*
3. *There are no simplices of width $w \geq 5$,*

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**Conjecture (Haase-Ziegler, 2000)**

The above list is complete. That is, there are no empty $4$-simplices of width $> 2$ and determinant $> 179$.

**Theorem (I.V.-Santos, 2018)**

This conjecture is true.

# Part I of the classification

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### Theorem 1 (I.V.-Santos, 2018)

There is no hollow $4$-simplex of width $> 2$ with determinant greater than $5058$.

### Theorem 2 (I.V.-Santos, 2018)

Up to determinant $\leq 7600$, all empty $4$-simplices of width larger than two have determinant in $[41, 179]$ and are as described explicitly by Haase and Ziegler.

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Indeed, no new simplex of width greater than two appear in the computations, so **the conjecture of Haase and Ziegler is true**.

# Hollow 3 bodies

We get a nice upper bound for the volume of hollow 3-bodies:

**Theorem (I.V.-Santos, 2018)**

*Let $w > 2.155$. Then, the following statements hold for any lattice-free convex body $K$ in dimension three of width at least $w$:*

1. $Vol(K) \leq \frac{3w^3}{4(w-(1+2/\sqrt{3}))}$ *if* $w \leq 2.427$,
2. $Vol(K) \leq \frac{8w^3}{(w-1)^3}$ *if* $w \geq 2.427$.

In particular if you restrict to lattice polytopes you get part (a) of Proposition 2 of AKW for a hollow 3 polytope $Q$:

$$\mathrm{vol}(Q) \leq 27$$

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**(Almost) Theorem 3 (Barile, Bernardi, Borisov and Kantor, 2011)**

All empty $4$-simplices that project to hollow $3$-polytopes belong to the $1 + 1 + 29$ families of Mori-Morrison-Morrison (1988), all of which have width one or two.

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But unfortunately,

Theorem 3 is only true for $4$-simplices of prime volume.
With non-prime volume another $0+1+23$ families arise (I.V.-Santos, 2017+).

# Classifying as projections

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*For each $d$, all except finitely many hollow $d$-polytopes (in particular, empty $d$-simplices) project to a hollow polytope of dimension $< d$.*

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This theorem implies that we can classify empty $4$-simplices attending at which dimension they project to:

- In dimension $3$ you can look at classification of hollow polytopes this way.
- Sporadic empty $4$-simplices are the finitely many simplices that do not project to a hollow polytope of dimension $\leq 3$.
- Empty $4$-simplices of width one project to hollow $1$-polytopes

# Classification of empty $4$-simplices

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**Main Theorem (I.V.-Santos, '18+)**

All except finitely many empty 4-simplices belong to one of the following cases:

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- $23$ additional $1$-parameter families that project to $23$ "non-primitive" hollow $3$-polytopes.

# Sporadic empty $4$-simplices

**Theorem 4 (I.V.-Santos, 2018+)**

Let $P$ be an empty $4$-simplex of width two and which do not project to a hollow $3$-polytope. Then $\mathrm{Vol}(P) \leq 7600$.

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**Theorem 4 (I.V.-Santos, 2018+)**

Let $P$ be an empty $4$-simplex of width two and which do not project to a hollow $3$-polytope. Then $\mathrm{Vol}(P) \leq 7600$.

This theorem implies that no *sporadic* simplices will appear with volume greater than $7600$.

**Theorem (I.V.-Santos, '18+)**

*There are exactly 2461 (classes of) empty $4$-simplices that do not belong to any of the infinite families shown in the theorem before. These empty $4$-simplices correspond to those that do not project to a hollow polytope of dimensions $d \in \{1, 2, 3\}$. Their determinants range from 24 to 419.*

- Mori et al. present some number of sporadic simplices for prime volume that are wrong due to some calculation mistakes

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# Thanks for your attention!
# And thank a lot Akiyoshi and Hibi


1 O. Iglesias Valiño and F. Santos, Classification of empty lattice $4$-simplices of width larger than two.
To be published in TAMS

2 O. Iglesias Valiño and F. Santos, The complete classification of empty lattice $4$-simplices.
In preparation



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# Appendix

For people interested in Ehrhart polynomials:

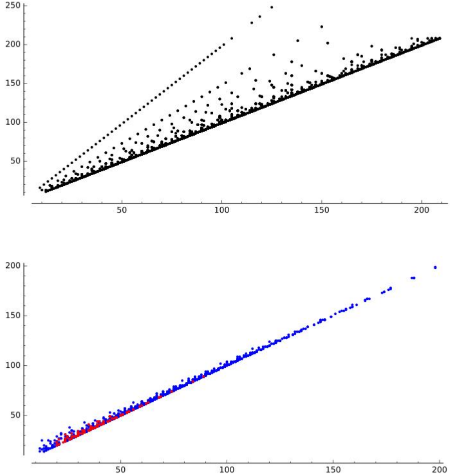


Figure : $h_3^*$ and $h_2^*$ coefficients for some families of simplices