The lecture hall cone as a toric deformation

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Lecture Hall Partitions

A Lecture Hall partition is a finite sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0.$$

The set $\mathbf{L}_n$ of Lecture Hall partitions is the set of lattice points in the Lecture Hall cone, which is the cone over the simplex with vertices

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n-1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n-1 \\ n-2 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} n \\ n-1 \\ n-2 \\ \vdots \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ n-1 \\ n-2 \\ \vdots \\ 2 \\ 1 \end{pmatrix}.$$

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The Lecture Hall Theorem

The original motivation for studying Lecture Hall Paritions is the Lecture Hall theorem. For a lecture hall partition λ let

$$|\lambda|_o := \lambda_1 + \lambda_3 + \lambda_5 + \cdots$$
$$|\lambda|_e := \lambda_2 + \lambda_4 + \lambda_6 + \cdots.$$

It holds that $$\sum_{\lambda \in L_n} q_1^{|\lambda|_o} q_2^{|\lambda|_e} = \prod_{i=1}^{n} \frac{1}{1 - q_1^i q_2^{i-1}}.$$

This is a rather strange specialization of the multivariate Ehrhart series. The multivariate Ehrhart series itself does not factor like this.

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Example: $n = 3$

For $n = 3$, a Hilbert basis of $\mathbf{L}_n$ is given by $v_1 := (1, 0, 0), v_2 := (2, 1, 0), v_3 := (3, 2, 0)$ and $v_4 := (3, 2, 1)$. There is one relation: $2v_2 = v_1 + v_3$. Thus the multigraded Hilbert series is

$$H(q_1, q_2, q_3) = \frac{1 - q_1^4 q_2^2}{(1 - q_1)(1 - q_1^2 q_2)(1 - q_1^3 q_2^2)(1 - q_1^3 q_2^2 q_3)}.$$

Specializing to $|\lambda|_o$ and $|\lambda|_e$ amounts to setting $q_3 = q_1$. This yields

$$\begin{aligned} H(q_1, q_2, q_1) &= \frac{1 - q_1^4 q_2^2}{(1 - q_1)(1 - q_1^2 q_2)(1 - q_1^3 q_2^2)(1 - q_1^3 q_2^2 q_1)} \\ &= \frac{1}{(1 - q_1)(1 - q_1^2 q_2)(1 - q_1^3 q_2^2)}, \end{aligned}$$

as predicted by the LH Theorem.

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There are a number of proofs of the LH Theorem, but it is still not considered to be well understood:

> *[...], Theorem 1.2 is hardly understood at all. This is in spite of the fact that by now there are many proofs, including those of Bousquet-Mélou and Eriksson [8–10], Andrews [1], Yee [55,56], Andrews, Paule, Riese, and Strehl [3], Eriksen [31], and Bradford et al. [11]. We have also contributed to the collection of proofs with co-authors Corteel [25], Corteel and Lee [20], Andrews and Corteel [2], Bright [15], and, most recently, Corteel and Lovejoy [23].*

C.D. Savage, “The mathematics of lecture hall partitions”, JCTA, 2016.

The Lecture Hall Theorem

Example $n = 3$, continued

Let's have another look at the example:

$$H(q_1, q_2, q_1) = \frac{1 - q_1^4 q_2^2}{(1-q_1)(1-q_1^2 q_2)(1-q_1^3 q_2^2)(1-q_1^3 q_2^2 q_1)}$$
$$= \frac{1}{(1-q_1)(1-q_1^2 q_2)(1-q_1^3 q_2^2)},$$

Observations:

- This works because the last generator has the same $\mathbb{Z}^2$ degree as the relation among the other generators.
- The Hilbert series looks like the Hilbert series of a polynomial ring with an unusual $\mathbb{Z}^2$ grading.

This leads to an idea...

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Initial Subalgebras

Some algebraic background

- Let $S_n := \mathbb{k}[y_1, \ldots, y_n]$ be a polynomial ring over some field.
- $p_1, \ldots, p_r \in S_n$ polynomials
- $\prec$: a term order on S_n.

Definition

Let $A := \mathbb{k}[p_1, \ldots, p_r]$ be the subalgebra generated by the $p_1, \ldots, p_r$. The inital subalgebra of A is

$$\mathrm{In}_\prec(A) := \mathrm{Span}_\mathbb{k}(\mathrm{In}_\prec(p) \mid p \in A)$$

In general we have that $\mathbb{k}[\mathrm{In}_\prec(p_1), \ldots, \mathrm{In}_\prec(p_r)] \subseteq \mathrm{In}_\prec(A)$. The $p_1, \ldots, p_r$ are called a SAGBI Basis if equality holds.

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Initial Subalgebras

Some algebraic background

Initial subalgebras behave similar to initial ideals, and SAGBI bases correspond to Gröbner bases.

Here we only need the following:

Lemma (Conca, Herzog, Trung, Valla 1997)

If S_n is graded and $A \subseteq S_n$ is a graded subalgebra, then the Hilbert series of A and $\mathrm{In}_{\prec}(A)$ coincide.

The Lecture Hall Theorem

Example $n = 3$, continued

Hilbert basis of $\mathbf{L}_3$: $(1,0,0),(2,1,0),(3,2,0),(3,2,1) \in \mathbb{Z}^3$

- Let $S_3 := \Bbbk[y_1, y_2, y_3]$ with grading $\deg y_1 := \deg y_3 := (1,0)$ and $\deg y_2 := (0,1)$.
- Ehrhart ring: $A := \Bbbk[y_1, y_1^2 y_2, y_1^3 y_2^2, y_1^3 y_2^2 y_3] \subset S_3$.
- Instead, we consider $\tilde{A} := \Bbbk[y_1, y_1^2 y_2, y_1^3 y_2^2 + y_1^2 y_2^2 y_3, y_1^3 y_2^2 y_3] \subset S_3$.
- Fact: $A = \mathrm{In}_\prec(\tilde{A})$, and thus their Hilbert series coincide. (Here, $\prec$ is any order with $y_3 \prec y_1$.)
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- Hence, $\tilde{A} = \Bbbk[y_1, y_1^2 y_2, y_1^3 y_2^2 + y_1^2 y_2^2 y_3]$. The generators are algebracially independent, hence this is a polynomial ring and the LH theorem follows (for $n = 3$).

Next, we try to guess similar polynomials for general n.

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The Lecture Hall Polynomials

Let $S_n := \Bbbk[y_1, y_2, \ldots, y_n]$. We consider the following $\mathbb{Z}^2$-grading on S_n:

$$\deg y_i := \begin{cases} (0,1) & \text{if } i \text{ is even,} \\ (1,0) & \text{if } i \text{ is odd.} \end{cases}$$

The Lecture Hall Polynomials

For a sequence of polynomials $\mathbf{P} := P_1, P_2, \ldots$ in S_n we define an infinite matrix $M(\mathbf{P})$ by setting

$$M(\mathbf{P})_{i,j} := \begin{cases} -P_{j-i+1} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, $M(\mathbf{P})$ looks as follows:

$$M(\mathbf{P}) = \begin{pmatrix} -P_1 & -P_2 & -P_3 & -P_4 & \ldots \\ 0 & -P_1 & -P_2 & -P_3 & \ldots \\ 0 & 0 & -P_1 & -P_2 & \ldots \\ 0 & 0 & 0 & -P_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(This is a Toeplitz matrix.)

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Notation:

- For $A, B \subset \mathbb{N}$ let $\Delta_B^A(M(\mathbf{P}))$ be the submatrix using rows in A and columns in B.
- For $i \in \mathbb{N}$ let

$$\mathcal{E}_i(\mathbf{P}) := -\det \Delta^{\{1,\ldots,\lceil i/2 \rceil\}}_{\{\lfloor i/2 \rfloor,\ldots,i\}}(M(\mathbf{P})).$$

In other words: $\mathcal{E}_i(\mathbf{P})$ is defined using the maximal top-aligned square submatrix of $M(\mathbf{P})$, whose top right corner is $-P_i$ and which does not contain any of the zeros of $M(\mathbf{P})$.

The Lecture Hall Polynomials

$$M(\mathbf{P})_{i,j} := \begin{cases} -P_{j-i+1} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

Notation:

- For $A, B \subset \mathbb{N}$ let $\Delta_B^A(M(\mathbf{P}))$ be the submatrix using rows in A and columns in B.
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The Lecture Hall polynomials are those Laurent polynomials $\ell_1, \ell_2, \ldots, \ell_n \in \mathrm{Quot}(S_n)$, such that

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Easy observations:

- Each ℓ_i is a Laurent polynomial, has coefficients in $\mathbb{Z}$, and is homogeneous of degree $(i, i-1)$.
- For each $i \geq 0$, the i-th Lecture Hall polynomial ℓ_i depends only on the variables $y_1, \ldots, y_i$, and it is non-constant as a function of y_i.

Conjecture

Each Lecture Hall polynomial ℓ_i is in fact a polynomial.

We verified this conjecture for $i \leq 12$. This is a purely combinatorial conjecture.

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Corollary

The algebra $A_n := \mathbb{Q}[\ell_1, \ldots, \ell_n] \subset \operatorname{Quot}(S_n)$ is isomorphic to a $\mathbb{Z}^2$-graded polynomial ring. Its Hilbert series equals

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This is the right-hand side of the LH theorem.

$\operatorname{In}_\prec(A_n)$ is supposed to be the Ehrhart ring. But the latter has 2^{n-1} generators, while A_n has only n. So the ℓ_n cannot be a SAGBI basis, and thus we need extra generators.

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A candidate for the SAGBI basis

Definition

For a finite set $S \subseteq \mathbb{N}, S \neq \emptyset$ let

$$\ell_S := -\det \Delta^{[\#S]}_{S+1}(M(\mathbf{P}_{\mathrm{LH}})),$$

where $S+1 := \{\, s+1 \mid s \in S \,\}$. In other words, ℓ_S is the negative of the minor of $M(\mathbf{P}_{\mathrm{LH}})$ using $\#S$ many top rows and the columns in $S+1$. In addition, we set $\ell_\emptyset := \ell_1$.

Note that $\ell_i = \ell_{\{\, i-1 \,\}}$ for $i \in \mathbb{N}$, and that
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Our main conjecture

Let $\prec$ be the degree-lexicographic term order on $S_n := \mathbb{Q}[y_1, \ldots, y_n]$ with $y_1 \succ y_2 \succ \ldots \succ y_n$, and let $A_n = \mathbb{Q}[\ell_1, \ldots, \ell_n]$.

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Assume that all the ℓ_i are polynomials. Then:

- This implies the LH theorem.
- Also, it implies that $\{ \ell_S \mid S \subseteq [n-1] \}$ is a SAGBI basis for A_n
- I verified the conjecture for $n \leq 12$.
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