Levelness of Order Polytopes

joint with Christian Haase and Akiyoshi Tsuchiya, see [HKT].

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Outline

- Let P be a lattice d-polytope.

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- We define the *cone over P* as
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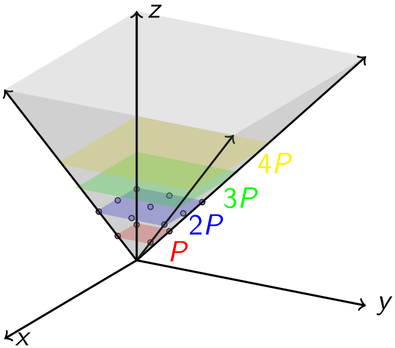

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- We set $C_{\mathbb{Z}}(P) := \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}$.
- This gives rise to the *semigroup algebra*

$$\Bbbk[P] := \Bbbk[C_{\mathbb{Z}}(P)] := \Bbbk[\boldsymbol{x}^{\boldsymbol{p}} \cdot y^m : (\boldsymbol{p}, m) \in C_{\mathbb{Z}}(P)].$$

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If $P = [0,1]^2$, then $\deg P = \deg(1+z) = 1$. P has codegree $2 = 3 - 1$. $[0,1]^2$ has no interior lattice points, but $[0,2]^2$ has one.

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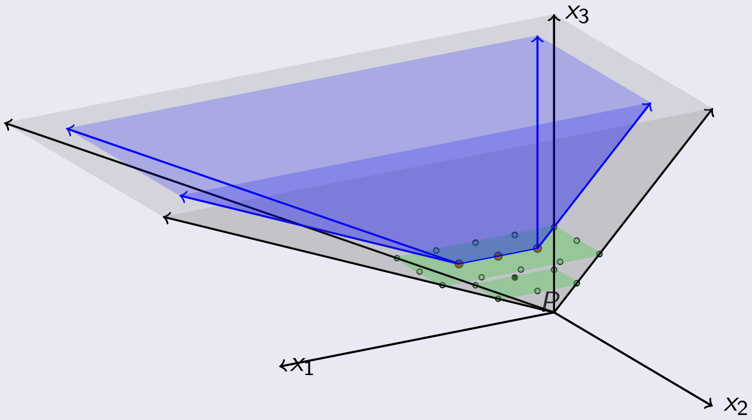
Definition

We say that P is *level* if the $\Bbbk[P]-$module $k[P^\circ]$ is generated by elements of the same degree. We say that P is *Gorenstein* if, moreover, there is a unique generator of minimal degree.

Example

Let $P = [0, 2] \times [0, 1]$. P is level and $\Bbbk[P^\circ]$ is generated by $(1, 1, 2)$, $(2, 1, 2)$, and $(3, 1, 2)$.



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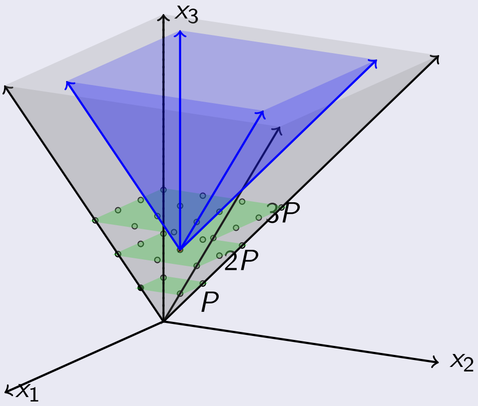

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Example

Let $(\Pi, \leq) = (2^{\{1,2,3\}}, \subset)$. Then the Hasse diagram is given by

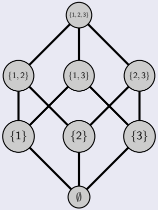

Figure: The Hasse diagram of $(2^{\{1,2,3\}}, \subset)$.

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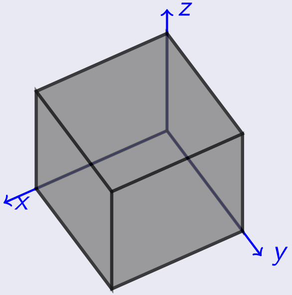
Definition

The *order polytope* $\mathcal{O}(\Pi)$ of a finite poset Π is the subset of $\mathbb{R}^{\Pi} = \{f : \Pi \to \mathbb{R}\}$ defined by

$$\begin{aligned} 0 \leq f(i) \leq 1 & \qquad \text{for all } i \in \Pi, \\ f(i) \leq f(j) & \qquad \text{if } i \leq_{\Pi} j. \end{aligned}$$

Example



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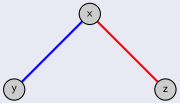

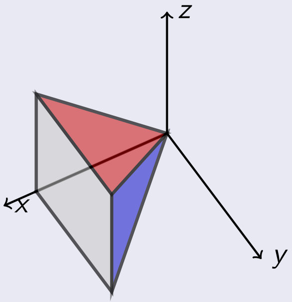

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Remark

On height k, $C_{\mathbb{Z}}(P^\circ) \cap \mathbb{R}^{d+1}$ is described by

$$\begin{aligned} 1 \leq f(i) \leq k-1 & \quad \text{for all } i \in \Pi, \\ f(i) \leq f(j) - 1 & \quad \text{if } i \lessdot j. \end{aligned}$$

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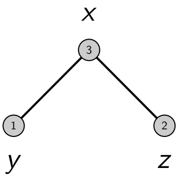

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Remark

A poset Π is Gorenstein if and only if every maximal chain has the same length, since we then have a unique interior lattice point in $(\operatorname{codeg} P)P$ and this point has distance 1 to all facets.

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- We will state a characterization of levelness in terms of weighted digraphs.

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Definition

Given a poset Π, we define the poset $\overline{\Pi} = (\Pi \cup \{\infty\}, \leq_{\overline{\Pi}})$, where

$$i <_{\overline{\Pi}} j :\Longleftrightarrow \begin{cases} j = \infty \text{ and } i \in \Pi, \\ i <_{\Pi} j. \end{cases}$$

Similarly, we define $\underline{\Pi} = (\Pi \cup \{-\infty\}, \leq_{\underline{\Pi}})$, where

$$i <_{\underline{\Pi}} j :\Longleftrightarrow \begin{cases} i = -\infty \text{ and } j \in \Pi, \\ i <_{\Pi} j. \end{cases}$$

Example

Let again $\Pi = y \lessdot x \gtrdot z$. Then $\overline{\Pi}$ is depicted in Figure 4.

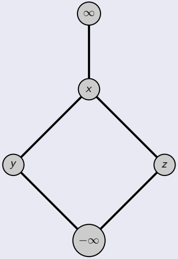

Figure: Figure 4.

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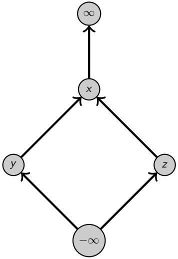

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- Now we can pick a set of edges Π' and add down edges of weight $+1$. The associated graph is denoted $\Gamma(\Pi')$.

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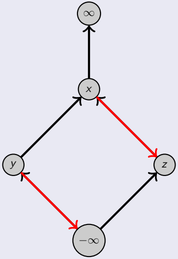
Example

Let $\Pi' = \{-\infty \lessdot y, z \lessdot x\}$.



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- This procedure gives rise to an integer point in $C_{\mathbb{Z}}(P^\circ)$ satisfying $f(i) + 1 = f(j)$ for all $i \lessdot j \in \Pi'$.

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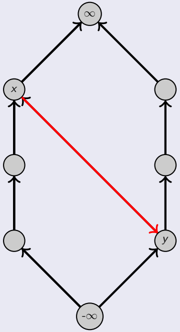
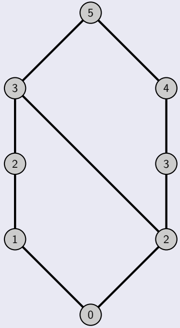
- This procedure gives rise to an integer point in $C_{\mathbb{Z}}(P^\circ)$ satisfying $f(i) + 1 = f(j)$ for all $i \lessdot j \in \Pi'$.
- Both can be done using the Bellman–Ford algorithm in polynomial time, to be precise, in $O(\#V \cdot \#E)$.

Example

Let $\Pi' = \{y \lessdot x\}$. Then $\Gamma(\Pi')$ is illustrated below, along with the integer point returned by the Bellman–Ford algorithm.



(a) $\operatorname{codeg}\Pi = 4$.



(b) Point on $y + 1 = x$.

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Theorem

A poset Π is level if and only if for all Π' such that $\Gamma(\Pi')$ does not contain a negative cycle, the graph $\Gamma(\Pi' \cup \{\text{longest chains}\})$ contains no negative cycle.

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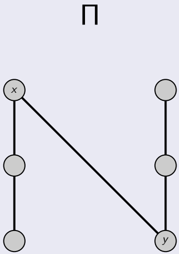
Remark

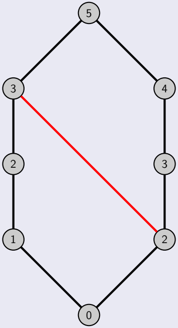
- If $\Gamma(\Pi')$ does not contain a negative cycle $\Leftrightarrow$ point on face $\bigcap_{i \lessdot j \in \Pi'} \{x_i + 1 = x_j\}$.
- No negative cycle in $\Gamma(\Pi' \cup \{\text{longest chains}\})$ $\Leftrightarrow$ there is a point on height codeg $\mathcal{O}(\Pi)$ on the same face.

Example

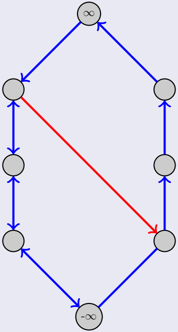
A non-level poset.



(a) codeg $\Pi = 4$.



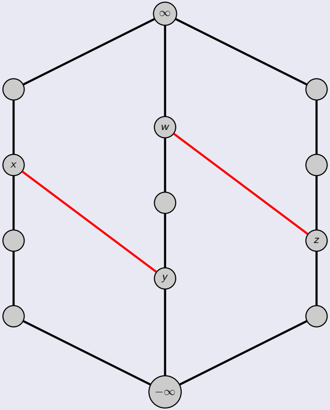
(b) Point on $y + 1 = x$.



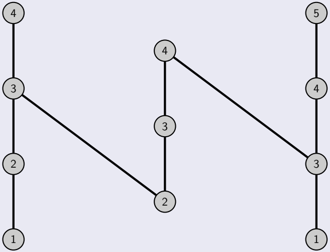
(c) A negative cycle in $\Gamma(\Pi' \cup \{\text{longest chains}\})$.

Example

A non-level poset, where Π' needs to contain at least two edges.



(a) Fink's poset.



(b) A point on the face $\{y + 1 = x\} \cap \{z + 1 = w\}$.

Corollary

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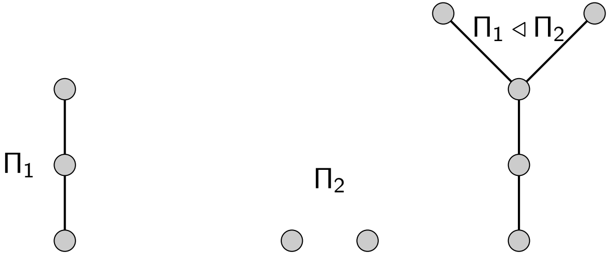

Figure: Ordinal sum of a chain of length 3 and an antichain of length 2.

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There are posets Π_1, Π_2 that have the same h^*-polynomial, but where Π_1 is level and Π_2 is not, see [Hib88].

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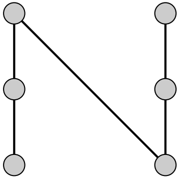
Theorem

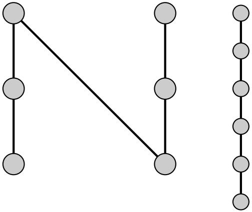
Let Π be a poset on d elements and $\Pi_1, \ldots, \Pi_m$ the connected components of Π. If each Π_i is level, then Π is level.

Theorem

Let Π be a poset on d elements and let C_s be the chain with s elements. Then the poset on the set $\Pi \cup C_s$, where elements from Π and C_s are incomparable, is level for all $s \geq d$.



(a) Non-level poset Π.



(b) Level poset $\Pi \cup C_6$.

Thanks for your attention!

Takayuki Hibi, *Level rings and algebras with straightening laws*, J. Algebra **117** (1988), no. 2, 343–362. MR 957445

Christian Haase, Florian Kohl, and Akiyoshi Tsuchiya, *Levelness of order polytopes*.

Mitsuhiro Miyazaki, *On the generators of the canonical module of a Hibi ring: a criterion of level property and the degrees of generators*, J. Algebra **480** (2017), 215–236. MR 3633306