Geometric structure of the Tesler polytopes

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2018

Tesler matrix

Definition[Tesler]

For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we say that a matrix $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbb{Z})$ is a Tesler matrix of hook sum α if :

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$$(a_{k,k} + a_{k,k+1} + ... + a_{k,n}) - (a_{1,k} + a_{2,k} + ... + a_{k-1,k}) = \alpha_k.$$

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Example

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

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Tesler polytope

Definition[Mészáros, Morales and Rhoades]

Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\mathbb{U}(n)_{\geq 0}$ be the set of n×n upper triangular matrices with non-negative real entries. Then the Tesler polytope of hook sum α is

$$\text{Tes}_n(\alpha) = \{A \in \mathbb{U}(n)_{\geq 0} : \text{k-th hook sum} = \alpha_k \text{ for } 1 \leq k \leq n\}$$

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Example

$n = 3$, $\alpha = (1, 2, 3)$ then

$$\text{Tes}_3(1,2,3) = \{A = \begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{bmatrix} \mid x_1 + x_2 + x_3 = \alpha_1,\ x_4 + x_5 - x_2 = \alpha_2,\ x_6 - x_3 - x_5 = \alpha_3,\ x_i \geq 0,\ 1 \leq i \leq 6\}$$

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The integer points inside the $\text{Tes}_n(\alpha)$ are exactly the Tesler matrices of hook sum α

Properties of Tesler polytopes

1. Unimodular in the hook sum subspace.
2. Simple
3. When $\alpha \in \mathbb{Z}^n_{>0}$, the face poset of $\mathrm{Tes}_n(\alpha) \cong$ Face poset of $\Delta_1 \times \Delta_2 \times \ldots \times \Delta_{n-1}$

Combinatorially Isomorphic polyhedra

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- The cone of feasible direction at a vertex of P is

$$\text{fcone}_v(P) = \{u \in \mathbb{R}^n \mid v + \delta u \in P \text{ for sufficiently small } \delta\}.$$

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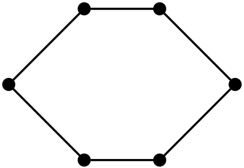

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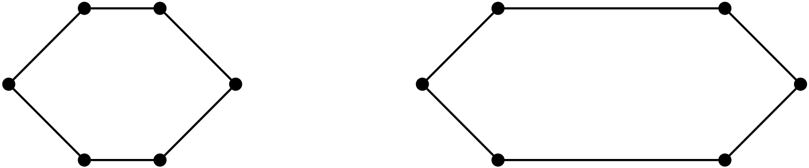

- (Castillo-Liu) For two simple polytopes $P = \mathrm{Conv}(v_1, \ldots, v_m)$ and $Q = \mathrm{Conv}(w_1, \ldots, w_m)$, we say Q is combinatorially isomorphic to P (Q is a deformation of P) if there exists $a_{i,j} \in \mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) satisfying $v_i - v_j = a_{i,j}(w_i - w_j)$ for each adjacent pair of vertices v_i, v_j of P and the corresponding vertices w_i, w_j of Q.
- P_1 is a deformation of P_2 if Σ_{P_2} is a refinement of Σ_{P_1}.

Theorem

Theorem (-,Liu)

For any $\alpha \in \mathbb{Z}^n_{>0}$, $\mathrm{Tes}_n(\alpha)$ *is combinatorially isomorphic to* $\mathrm{Tes}_n(1,...,1)$. *Furthermore, when some coordianates of* α *are zero,* $\mathrm{Tes}_n(\alpha)$ *is a deformation of* $\mathrm{Tes}_n(1,...,1)$.

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Has an application on alpha positivity of Tesler polytopes!

Mcmullen's formula

Theorem (Mcmullen's formula)

Let P be a d-dimensional polytope. Then there exists a funtion α such that the following is true :

$$|P \cap \mathbb{Z}^n| = \sum_{F \,:\, \text{face of } P} \alpha(F, P)\,\mathrm{Vol}_{dimF}(F).$$

Where $\alpha(F, P)$ only depends on the normal cone of P at F.

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Corollary

Let P be a polytope and $E_P = e_d t^d + \ldots + e_1 t + 1$ be it's Ehrhart polynomial. Then, $e_i = \sum_{F \text{ : i-th dimensional face of } P} \alpha(F,P)\,\mathrm{Vol}_i(F)$.

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But α is not unique!

- Berline-Vergne alpha (BV-alpha).
- Ring-Schürmann mu (RS-mu).

Reduction theorem

Theorem (Castillo-L, Reduction Theorem)

Suppose Ψ *is a function on indicator functions of rational cones* C *in* V *such that*

- Ψ *is a valuation(linear transformation) on the algebra of rational cones in* V
- *If a cone* C *contains a line, then* $\Psi([C]) = 0$.

Let P *and* Q *be two polytopes in* V *such that* Q *is a deformation of* P*. Then for any fixed* k*, if we set* $\alpha(F, P) = \Psi([\text{ncone}(F, P)/\ \text{lin}(F)])$ *and* $\alpha(F, P) > 0$ *for every* k*-dimensional face* F *of* P*, then* $\alpha(G, Q) > 0$ *for every* k*-dimensional face* G *of* Q*.*

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Corollary

If Q is a deformation of P and P is BV-alpha positive, then Q is also BV-alpha positive.

For fixed n, If $\mathrm{Tes}_n(1, ..., 1)$ is BV-alpha positive, then $\mathrm{Tes}_n(\alpha)$ is BV-alpha positive for any $\alpha \in \mathbb{Z}^n_{\geq 0}$.

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Conjecture (Morales) : $\text{Tes}_n(1, ..., 1)$ is Ehrhart positive.

Projections of Tesler Polytopes

Theorem

Let $\mathbb{U}(n)$ *be the set of nxn upper triangular matrices with non-negative entries and* $(x_{i,j}) \in \mathbb{U}(n)$. *Then for any* $(j_1, j_2, ..., j_{n-1}) \in [n] \times [n-1] \times [n-1] \times ... \times [2]$,
A map $\phi : \mathbb{U}(n) \longrightarrow \mathbb{U}(n)$ *defined by*

$$\phi((x_{i,j})) = (y_{i,j}) Where \quad y_{i,j} = \begin{cases} x_{i,j} & if\ j \neq j_i \\ 0 & if\ j = j_i \end{cases}$$

Defines a unimodular transformation between the hook sum space and $\mathbb{R}^{\binom{n}{2}}$.

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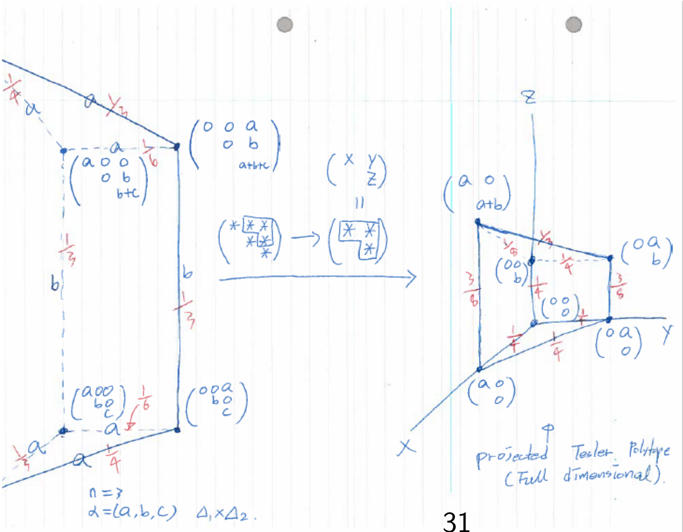
$$\begin{bmatrix} d_1 & x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} \\ & d_2 & x_{2,2} & \cdots & x_{2,n-1} \\ & & \ddots & \cdots & \vdots \\ & & & d_{n-1} & x_{n-1,n-1} \\ & & & & d_n \end{bmatrix} \longrightarrow \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} \\ & x_{2,2} & \cdots & x_{2,n-1} \\ & & \ddots & \vdots \\ & & & x_{n-1,n-1} \end{bmatrix}$$

Projection examples



BV-alpha positivity of some faces of the Tesler polytope

- BV-alpha for codim=2 face F. Let $C = \text{fcone}(F,P)/\operatorname{lin}(F) = \text{Cone}(u_1,u_2)$ where u_1 and u_2 form a basis for the orthogonal projection of $\mathbb{Z}^n$ to $\mathbb{R}^n/\operatorname{lin}(F)$. Then,

$$\alpha(F,P) = \frac{1}{4} + \frac{1}{12}\left(\frac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle} + \frac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle}\right).$$

- BV-alpha for codim=3 face F. Let $C = \text{fcone}(F,P)/\operatorname{lin}(F) = \text{Cone}(u_1,u_2,u_3)$. Where u_1 and u_2 form a basis for the orthogonal projection of $\mathbb{Z}^n$ to $\mathbb{R}^n/\operatorname{lin}(F)$. Then,

$$\alpha(F,P) = \tfrac{1}{8} + \tfrac{1}{24}\left(\tfrac{\langle u_1,u_2\rangle}{\langle u_1,u_1\rangle} + \tfrac{\langle u_1,u_2\rangle}{\langle u_2,u_2\rangle} + \tfrac{\langle u_1,u_3\rangle}{\langle u_1,u_1\rangle} + \tfrac{\langle u_1,u_3\rangle}{\langle u_3,u_3\rangle} + \tfrac{\langle u_2,u_3\rangle}{\langle u_2,u_2\rangle} + \tfrac{\langle u_2,u_3\rangle}{\langle u_3,u_3\rangle}\right).$$

- codim=4 formula has way more than 1000 terms...

RS-mu positivity of some faces of the Tesler polyrope

Let F be a codimension 2 face of $Tes_3(1,1,1)$.

$$\mu(F) = 1 - (1 - a - b)\mu(Tes_3(1,1,1)) - \frac{1}{2}\mu(Facet)x2$$

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$$\mu(F) = b$$

Flow Polytopes and Tesler polytopes

Definition

For any $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$, the flow polytope $\text{Flow}_n(\alpha)$ of complete graph K_{n+1} with net flow α_i on vertex i for $i = 1, 2, ..., n$ and $-\sum_{i=1}^{n} \alpha_i$ on $n+1$ is the set of functions $f : E \to \mathbb{R}_{\geq 0}$ where E is the edge set of K_{n+1} such that $\sum_{j>s} f(k,j) - \sum_{i<k} f(i,k) = \alpha_k$.

Theorem (Mészáros, Morales, Rhoades.)

for $\alpha \in \mathbb{Z}^n_{\geq 0}$, $\text{Flow}_n(\alpha) \cong \text{Tes}_n(\alpha)$.

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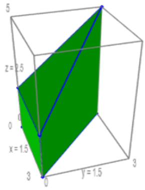
3. Wall := Hyperplane generated by n-1 elements of B_n^+
4. Chamber := a connected component of $\mathrm{Cone}(B_n^+) - \bigcup_{W \,:\, Wall} W$

Flow polytopes in different chambers



(3,2,1)

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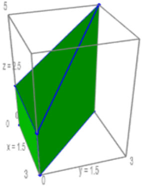

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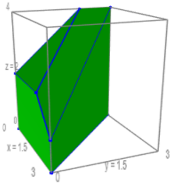

(3,2,-1)

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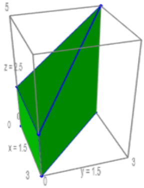

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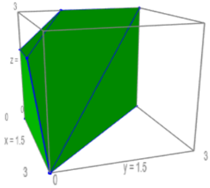
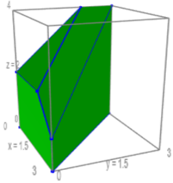


(3,2,-2)



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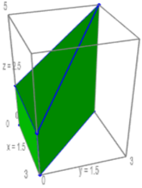

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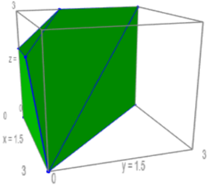

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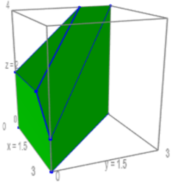

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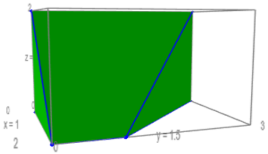

(3,2,-3)

Some questions

1. $\alpha(F, \mathrm{Tes}_n(1, ..., 1)) = \mu(F, \mathrm{Tes}_n(1, ..., 1))$?
2. If α_1 and α_2 is from the same chamber, $\mathrm{Flow}_n(\alpha_1)$ and $\mathrm{Flow}_n(\alpha_2)$ are combinatorially isomorphic?
3. For fixed n, which of the chambers give polytopes that are deformations of $\mathrm{Tes}_n(1, ..., 1)$?

Vertices of the Tesler polytopes

For any *nxn* matrix $A = (a_{i,j})$, define the support map $s(A) = (s_{i,j})$ where

$$s_{i,j} = \begin{cases} 1, & \textit{if } a_{i,j} \neq 0 \\ 0, & \textit{if } a_{i,j} = 0 \end{cases}$$

Theorem (Mészáros, Morales and Rhoades)

For $\alpha \in \mathbb{N}^n$,

- $\text{Vert}(\text{Tes}_n(\alpha)) = \{A \ : \ A \textit{ is a permutation Tesler matrix}\}$ *Where permutation Tesler matrices are the nxn Tesler matrices with exactly one nonzero entry in each rows.*
- *Two vertices v and w of* $\text{Tes}_n(\alpha)$ *are adjacent iff* $s(v)$ *can be obtained from* $s(w)$ *by moving 1 in a row to a different column in the same row.*

Idea of the proof by example

$\alpha = (3, 2, 3, 2, 1)$

$$v = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ & 0 & 5 & 0 & 0 \\ & & 8 & 0 & 0 \\ & & & 0 & 2 \\ & & & & 3 \end{bmatrix} \qquad w = \begin{bmatrix} 0 & 0 & 0 & 3 & 0 \\ & 0 & 2 & 0 & 0 \\ & & 5 & 0 & 0 \\ & & & 0 & 5 \\ & & & & 6 \end{bmatrix}$$

$$w - v = \begin{bmatrix} 0 & -3 & 0 & 3 & 0 \\ & 0 & -3 & 0 & 0 \\ & & -3 & 0 & 0 \\ & & & 0 & 3 \\ & & & & 3 \end{bmatrix} = 3 \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ & 0 & -1 & 0 & 0 \\ & & -1 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 1 \end{bmatrix}$$

In general, $w - v = p * M$ where p is some positive integer and M is a matrix consisting of $0, 1, -1$'s (M is in fact an edge direction). Changing α only changes p but doesn't affect M.

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