Gram's relation for cone valuations

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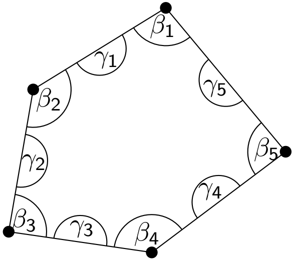
A fundamental theorem of Euclidean geometry:



Theorem

For a polygon with n vertices, it holds

$$\sum_{i=1}^{n} \beta_i - n\pi + 2\pi = 0 \sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \gamma_i + 2\pi = 0$$

Cone angles

A cone angle α assigns to each polyhedral cone C in $\mathbb{R}^d$ a real number, such that

- α is a valuation ($\alpha(C \cup C') = \alpha(C) + \alpha(C') - \alpha(C \cap C')$ whenever $C \cup C'$ is convex),
- α is simple ($\alpha(C) = 0$, whenever $\dim C < d$),
- α is normalized ($\alpha(\mathbb{R}^d) = 1$).

Example

- *Take* $\nu(C) := \frac{\mathrm{vol}(C \cap B_d)}{\mathrm{vol}\, B_d}$ *and this gives the usual angle.*

Main Theorem

Theorem (1)

Let $P \subseteq \mathbb{R}^d$ be a polytope and α be a cone angle. For $i = 0, \ldots, d-1$, let

$$\widehat{\alpha}_i(P) := \sum_{\substack{F \subseteq P, \\ \dim F = i}} \alpha(\widehat{A}(F, P)),$$

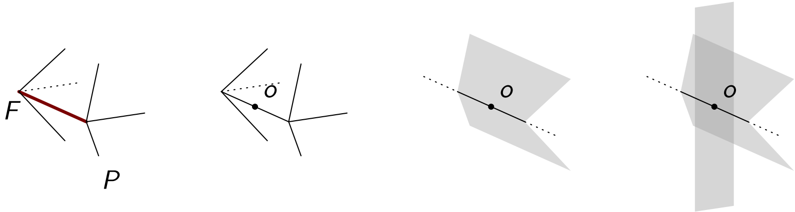
where $\widehat{A}(F, P)$ is the interior cone at F. Then the only linear relation among these numbers is the Gram-relation:

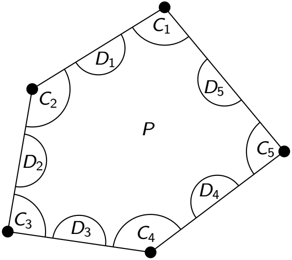
$$\sum_{i=0}^{d-1} (-1)^i \widehat{\alpha}_i(P) = (-1)^{d+1}.$$

Interior angles

Let F be a face of a polytope P. Define the interior cone $\widehat{A}(F,P)$ by the following procedure: $\widehat{A}(F,P) = \text{cone}(P-q) + \text{cone}(P-q)^{\perp}$.

- Move the origin to a point in the relative interior of F.
- Take the conical hull of P.
- (Minkowski-) add the dual of the affine hull (to make the cone full dimensional if P is not full-dimensional).





Example

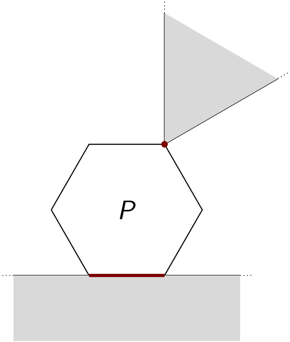
- $\widehat{\alpha}_0(P) = \alpha(C_1) + \alpha(C_2) + \alpha(C_3) + \alpha(C_4) + \alpha(C_5)$
- $\widehat{\alpha}_1(P) = \alpha(D_1) + \alpha(D_2) + \alpha(D_3) + \alpha(D_4) + \alpha(D_5)$

and $\widehat{\alpha}_0(P) - \widehat{\alpha}_1(P) = -1$. *If* $\alpha = \nu$, *we recover the original statement after multiplying with* 2π.

Exterior angles

Similarly, we define exterior cones $\breve{A}(F, P)$ by taking the normal cone and adding the orthogonal complement:

$$\breve{A}(F, P) := N_F P + (N_F P)^{\perp}.$$



Exterior angles

Theorem (1b)

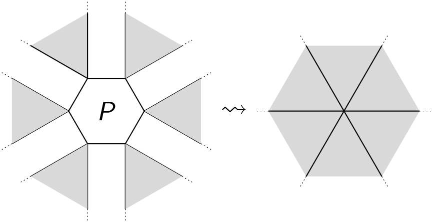
Let $P \subseteq \mathbb{R}^d$ be a polytope and α be a cone angle. For $i = 0, \ldots, d-1$, let

$$\breve{\alpha}_i(P) := \sum_{\substack{F \subseteq P, \\ \dim F = i}} \alpha(\breve{A}(F, P)) .$$

Then the only linear relation among these numbers is:

$$\breve{\alpha}_0(P) = 1 .$$

Partial proof of 1b



The normal cones of the vertices of a polytope form the normal fan. This is a complete fan. Thus:

$$\widehat{\alpha}_0(P) = \sum_{v \in \text{vert}(P)} \alpha(\check{A}(v,P)) = \sum_{v \in \text{vert}(P)} \alpha(N_v P) = \alpha(\mathbb{R}^d) = 1\,.$$

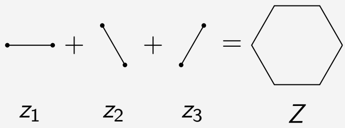
Zonotopes

To prove, that there is only one linear relation, we need to find a set of polytopes, whose $\widehat{\alpha}_i$'s (resp. $\breve{\alpha}_i$'s) span the single hyperplane defined by the Gram-relation (resp. $\breve{\alpha}_0(P) = 1$).

Definition

A zonotope Z is the Minkowski-sum of line segments $z_1, \ldots, z_n$:
$Z = z_1 + \cdots + z_n \in \mathbb{R}^d$.

Example



Zonotopes II

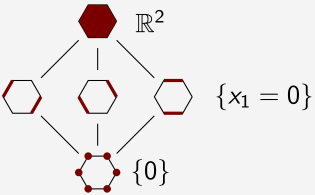
To every zonotope Z with face lattice $\mathcal{F}(Z)$ we can associate the lattice of non-empty faces

$$\mathcal{F}_+(Z) := \mathcal{F}(Z) \setminus \{\emptyset\}$$

and the lattice of flats $\mathcal{L}(Z)$:

$$\mathcal{L}(Z) := (\{\text{aff}(F - q) : F \in \mathcal{F}_+(Z), q \in F\}, L_1 \preceq L_2 :\Longleftrightarrow L_1 \subseteq L_2)$$

Example



This gives a map $L : \mathcal{F}_+(Z) \to \mathcal{L}(Z)$, $L(F) = \text{aff}(F - q)$ for some $q \in F$.

Incidence Algebra

The incidence algebra $\mathcal{I}(\mathcal{P})$ of a poset $\mathcal{P} = (\mathcal{P}, \preceq)$ is the vector space of all functions $h : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, such that $h(a, c) = 0$ whenever $a \not\preceq c$ and with multiplication

$$(g * h)(a, c) \ = \ \sum_{a \preceq b \preceq c} g(a, b)h(b, c)$$

for $g, h \in \mathcal{I}(\mathcal{P})$.

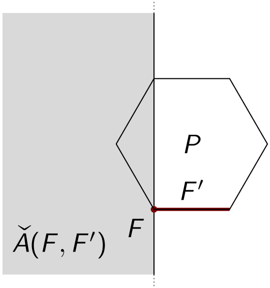
Example

- $\delta_{\mathcal{P}}(a, b) = 1$ *if* $a = b$, $\delta_{\mathcal{P}}(a, b) = 0$ *otherwise, is the unit of the multiplication.*
- $\zeta_{\mathcal{P}}(a, b) = 1$ *for* $a \preceq b$.
- $\zeta_{\mathcal{P}}$ *always has an inverse:* $\mu_{\mathcal{P}} = \zeta_{\mathcal{P}}^{-1}$.

Example

For a cone angle α both $\widehat{\alpha}$, $\check{\alpha}$ are in $\mathcal{I}(\mathcal{F}_+(P))$.

Let F, F' be two non-empty faces. Define $\widehat{\alpha}(F, F') = \alpha(\widehat{A}(F, F'))$ and $\check{\alpha}(F, F') = \alpha(\check{A}(F, F'))$.



Main Theorem II

Theorem (2)

For any cone angle α and any d-zonotope Z, we have

$$\widehat{\alpha}_i(Z) = |w_i^{\mathrm{co}}(\mathcal{L}(Z))|\,,$$
$$\check{\alpha}_i(Z) = W_i^{\mathrm{co}}(\mathcal{L}(Z))\,,$$

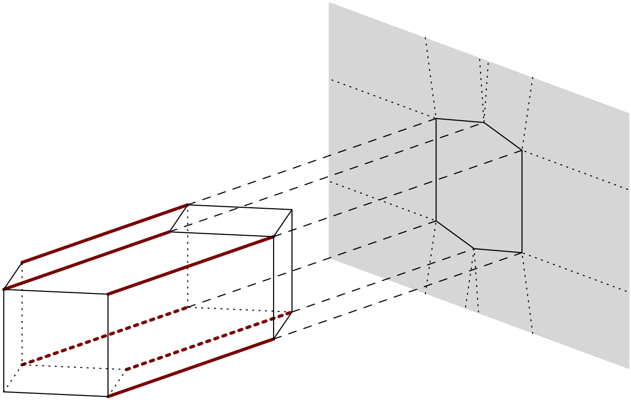
where w_i^{co}, w_i^{co} are the Co-Whitney numbers of first and second kind defined by:

$$w_i^{\mathrm{co}} = \sum_{M \in \mathcal{L}(Z)_i} \mu_{\mathcal{L}(Z)}(M, \mathbb{R}^d)\,,$$
$$W_i^{\mathrm{co}} = \sum_{M \in \mathcal{L}(Z)_i} \zeta_{\mathcal{L}(Z)}(\{0\}, M) = \sum_{M \in \mathcal{L}(Z)_i} 1\,.$$

Proof of Theorem 1.

Find d zonotopes where you can calculate the Co-Whitney numbers easily (uniform matroids). Then do linear algebra to show that the Co-Whitney numbers span the single hyperplane defined by the Gram-relation (resp. $\check{\alpha}_0(P) = 1$). □

How to prove theorem 2?

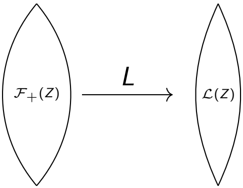


Thus $\sum_{F \in L^{-1}(M)} \breve{\alpha}(F, Z) = 1$ and summing over all M with $\operatorname{rk} M = i$ gives $\breve{\alpha}_i(Z) = W_i^{\mathrm{co}}(\mathcal{L}(Z))$.

How to prove theorem 2?

You can prove the other half of Theorem 2 directly using the valuation property and the theory of hyperplane arrangements.

It also follows from the fact, that $L : \mathcal{F}_+(Z) \to \mathcal{L}(Z)$ induces two adjoint maps L^* and L_*:



$$\mathcal{I}(\mathcal{F}_+(Z)) \xrightarrow{L^*} \mathcal{I}(\mathcal{L}(Z))$$

$$\mathcal{I}(\mathcal{F}_+(Z)) \xleftarrow{L_*} \mathcal{I}(\mathcal{L}(Z))$$

You can then show that $L^*(\widehat{\alpha}) = |\mu_{\mathcal{L}(Z)}|$ and $L^*(\check{\alpha}) = \zeta_{\mathcal{L}(Z)}$ for any cone angle α.

Thank you for your attention. Questions?