Transversal Valuated Matroids and Stiefel Presentations

Jorge Alberto Olarte

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joint work with Alex Fink

Freie Universität Berlin

The Classical Stiefel map

Definition

Let $\mathbb{K}$ be a field. The *(rational) Stiefel map* $\boldsymbol{\pi} : \mathbb{K}^{d\times n} \dashrightarrow \mathrm{Gr}_{\mathbb{K}}(n,d)$ sends a rank d matrix A to its row span in $\mathrm{Gr}_{\mathbb{K}}(d,n)$.

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Given a d dimensional subspace $L \in \mathrm{Gr}_{\mathbb{K}}(n,d)$, what is its preimage under the Stiefel map, $\boldsymbol{\pi}^{-1}(L)$?

Answer

Any matrix A such that its rows are d linearly independent vectors in L. The space of all such matrices is $\mathrm{GL}(d,\mathbb{K})$.

Transversal matroids

Definition

A *matroid* of n elements and rank d is a non empty set of bases $M \subseteq \binom{[n]}{d}$ such that for any $B_1, B_2 \in M$ and $i \in B_1 \setminus B_2$ there exists $j \in B_2$ such that $B_1 \cup \{j\} \setminus \{i\} \in M$.

Definition

Let $A_1, \ldots, A_d$ be subsets of $[n]$. A *transversal* is a subset $B \in \binom{[n]}{d}$ such that there is a bijection $\phi : \; B \to [d]$ such that $i \in A_{\phi(i)}$ for every $i \in B$. The set of transversals, if not empty, is a matroid. Any matroid that arrises this way is called a *transversal matroid*. The multiset $\{\!\{A_1, \ldots, A_d\}\!\}$ is called a *presentation* of the matroid.

Classical Plücker relations

Definition

The *Plücker embedding* $\iota : \mathrm{Gr}_{\mathbb{K}}(n,d) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$ sends an element $L \in \mathrm{Gr}_{\mathbb{K}}(n,d)$ to the maximal minors of A where A is such that $\boldsymbol{\pi}(A) = L$.

The image of the Plücker embedding is the projective variety given by the following equations in $\mathbb{P}^{\binom{n}{d}-1}$: for any $S \in \binom{[n]}{d-1}$ and $T \in \binom{[n]}{d+1}$,

$$\sum_{i \in T \setminus S} (-1)^i X_{S \cup \{i\}} X_{T \setminus \{i\}} = 0$$

A point $x \in \mathbb{R}^n$ is in L if and only if for any $T \subseteq [n]$ with $|T| = d+1$ it satisfies:

$$\sum_{i \in T} (-1)^i x_i \, \iota(L)_{T \setminus \{i\}} = 0$$

Valuated matroids

Recall the tropical semiring $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$ where $\mathbb{R} \cup \infty$ has the topology of $(0, 1]$. The k-dimensional tropical projective space $\mathbb{TP}^k$ is the space $\mathbb{T}^{k+1} \setminus \{(\infty, \ldots, \infty)\}$ quotiented by the equivalence relation $\sim$ given by $(a_0, \ldots, a_d) \sim (a_0 + \lambda, \ldots, a_d + \lambda)$ for any $\lambda \in \mathbb{T} \setminus \{\infty\}$.

Definition

A *valuated matroid* $\mathcal{V}$ is a point in $\mathbb{TP}^{\binom{n}{d}-1}$ that satisfies the tropical Plücker relations: for any $S \in \binom{[n]}{d-1}$ and $T \in \binom{[n]}{d+1}$, the minimum in

$$\min_{i \in T \setminus S} \mathcal{V}_{S \cup \{i\}} + \mathcal{V}_{T \setminus \{i\}}$$

is achieved twice. The space of valuated matroids in $\mathbb{TP}^{\binom{n}{d}-1}$ is called the *Dressian* $\mathrm{Dr}(n, d)$.

Tropical linear spaces

Theorem [Speyer (2008)]

Let $\mathcal{V}$ be a valuated matroid and let $x \in \mathbb{R}^n$. The set

$$M_x = \operatorname{argmin}_{B \in \binom{[n]}{d}} \left(\mathcal{V}_B - \sum_{i \in B} x_i \right)$$

is a matroid.

Note that if $x \sim y$ then $M_x = M_y$. Recall that an element i of a matroid is called a *loop* if it is not contained in any basis. It is called a *coloop* if it is contained in every basis.

Definition

Let $\mathcal{V}$ be a valuated matroid. The *tropical linear space* $L \subseteq \mathbb{TP}^{n-1}$ associated to $\mathcal{V}$ is the closure of

$$\{x \in \mathbb{R}^n \mid M_x \text{ has no loops}\} / \sim$$

Matroid polytopes

Let $e_1, \ldots, e_n$ be the cannonical basis of $\mathbb{R}^n$ and for $B \subseteq [n]$ let

$$e_B = \sum_{i \in B} e_i$$

Definition

The *matroid polytope* P_M of a matroid M is the convex hull of $\{e_B \mid B \in M\}$.

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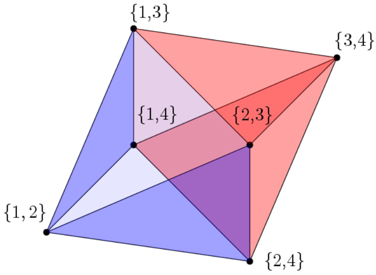
A valuated matroid $\mathcal{V}$, induces a regular subdivision of $\text{conv}(\{e_B \mid \mathcal{V}_B < \infty\})$ into matroid polytopes P_M. L is a polyhedral complex with cells

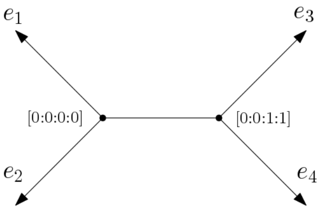
$$L_M = \{x \in L \mid M_x = M\}$$

Each L_M is orthogonal to P_M.

Example 1

Consider $\mathcal{V} = [0:0:0:0:0:1] \in \mathrm{Dr}(2,4)$.





The subdivision of the hypersimplex and the tropical linear space L induced by $\mathcal{V}$.

Tropical Stiefel map

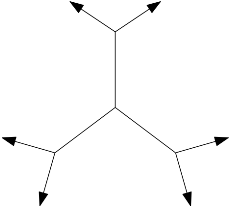
Given a matrix $A \in \mathbb{T}^{d\times n}$ and a subset $\{b_1, \ldots, b_d\} \in \binom{[n]}{d}$ the corresponding tropical maximal minor is

$$\min_{\sigma \in S_d} \left(\sum_{i=1}^{d} A_{b_i, b_{\sigma(i)}} \right)$$

Definition

The *tropical Stiefel map* $\pi : \mathbb{T}^{d\times n} \dashrightarrow \mathrm{Dr}(d,n)$ maps A to its tropical maximal minors, whenever A does not have all of its tropical maximal minors equal to ∞.

As opposed to the classical case, $\pi(\mathbb{T}_{\mathrm{tr}}^{d\times n}) \subsetneq \mathrm{Dr}(d,n)$. The tropical linear space corresponding to $\mathcal{V} \in \mathrm{Dr}(2,6)$ where $\mathcal{V}_{12} = \mathcal{V}_{34} = \mathcal{V}_{56} = \infty$ and zero elsewhere is not in the Stiefel image.



The Stiefel image

Definition

A *trasnversal valuated matroid* $\mathcal{V}$ is a valuated matroid such that $M_x(\mathcal{V})$ is transversal whenever $M_x(\mathcal{V})$ is coloop free.

Theorem, [Fink, Rincon (2015)]

Let $\mathcal{V} \in \mathrm{Dr}(d, n)$. If there exists a matrix A such that $\pi(A) = \mathcal{V}$, then $\mathcal{V}$ is a transversal valuated matroid.

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The converse is also true:

Theorem, [Fink, O. (2018+)]

Let $\mathcal{V}$ be a transversal valuated matroid. Then there exists a matrix A such that $\pi(A) = \mathcal{V}$.

Stiefel presentations

Question

Let $\mathcal{V}$ be a transversal valuated matroid. What is $\pi^{-1}(\mathcal{V})$?

Definition

A *Stiefel presentation* is a multiset of d points in L whose representatives form the rows of a matrix $A \in \pi^{-1}(\mathcal{V})$.

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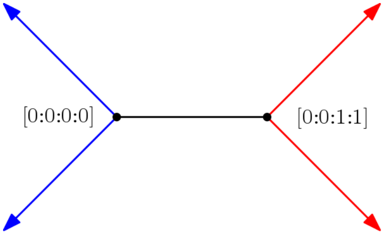
Presentations of $\mathcal{V} = [0:0:0:0:0:1]$

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & * & 1 \end{pmatrix}$$

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & 0 & 1 & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 1 & * \end{pmatrix}$$



Cyclic flats

Recall the following basic definitions from matroid theory:

Definition

Let M be a matroid on n elements of rank d. For a subset $F \subset [n]$:

- The *rank* of F is $\operatorname{rk}(F) := \max_{B \in M}(|B \cap F|)$
- The *corank* of F is $\operatorname{cork}(F) = d - \operatorname{rk}(F)$.
- F is called a *flat* if for every $i \in [n] \setminus F$ we have that $\operatorname{rk}(F \cup \{i\}) = \operatorname{rk}(F) + 1$
- F is called *cyclic* if for every $i \in F$ we have that $\operatorname{rk}(F \setminus \{i\}) = \operatorname{rk}(F)$

The set of cyclic flats form a lattice $\mathcal{CF}(M)$ with respect to inclusion.

Presentations of transversal matroids

Definition

Let τ_M be the function implicitly defined on $\mathcal{CF}(M)$ by the following equation

$$\sum_{F \subseteq F' \in \mathcal{CF}} \tau_M(F') = \mathrm{cork}(F)$$

Theorem, [Mason (1970)]

Let M be a transversal matroid. The multiset $\{\!\{[n] \setminus F_1, \dots, [n] \setminus F_d\}\!\}$, where $F_i \in \mathcal{CF}$ and each $[n] \setminus F_i$ appears with multiplicity $\tau_M(F_i)$, is the unique maximal presentation of M. Any presentation $\{\!\{[n] \setminus A_1, \dots, [n] \setminus A_d\}\!\}$ satisfy:

1. Each A_i is flat.
2. $A_i \supseteq F_i$ for each $i \in [d]$ (up to relabelling).
3. For each subset of indexes $I \subset [d]$ we have

$$\mathrm{cork}\left(\bigcap_{i \in I} A_i\right) \geq |I|$$

The multiset of distinguished apexes.

Definition

Let $\mathcal{V}$ be a transversal valuated matroid and L its tropical linear space. The multiset $\mathcal{A}(\mathcal{V})$ of *distinguished apexes* of $\mathcal{V}$ consists of points $x \in L$ such that x appears with multiplicity $\tau_{M_x(\mathcal{V})}(F)$ where $F \subseteq [n]$ are the coordinates of x equal to ∞.

Proposition

The function $P_M \to \tau_M(\emptyset)$ is a simple valuation on matroid polytopes.

Proposition

If $\mathcal{V}$ is of rank d, then $|\mathcal{A}(\mathcal{V})| = d$

The space of Stiefel presentations

The following is a generalization of Mason's theorem for transversal valuated matroids.

Theorem, [Fink, O. (2018+)]

Let $\mathcal{V}$ be a transversal valuated matroid. Then:

1. $\mathcal{A}(\mathcal{V})$ is a Stiefel presentation.
2. The space of all Stiefel presentations of $\mathcal{V}$ consists of fans centered at each of the distinguished apexes.
3. The fan over $x \in \mathcal{A}(\mathcal{V})$ only depends on $M_x(\mathcal{V})$.

Example 2

Let $\mathcal{V} \in \mathrm{Dr}(3,5)$ where $\mathcal{V}_{123} = 1$, $\mathcal{V}_{145} = \infty$ and 0 elsewhere. The three distinguished apexes are $x_1 = (0:0:0:0:0)$, $x_2 = (1:1:1:0:0)$ and $x_3 = (\infty:0:0:\infty:\infty)$.