Level algebras and lecture hall polytopes

(Joint with Florian Kohl)

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Algebraic and Geometric Combinatorics on Lattice Polytopes
Osaka University

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Overview

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Definition

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ be a sequence such that $s_i \in \mathbb{Z}_{\geq 1}$ for all i. The $\boldsymbol{s}$-*lecture hall partitions* are the set

$$L_n^{(\boldsymbol{s})} := \left\{ \lambda \in \mathbb{Z}^n \, : \, 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}.$$

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Example

Let $\boldsymbol{s} = (2,5,7)$. The partition $(1,4,6) \in L_3^{(2,5,7)}$ as $\frac{1}{2} \leq \frac{4}{5} \leq \frac{6}{7}$.

However, $(1,4,5) \notin L_3^{(2,5,7)}$ as $\frac{1}{2} \leq \frac{4}{5} > \frac{5}{7}$.

Inversion sequences and statistics

Given $\boldsymbol{s}$, the set of $\boldsymbol{s}$-*inversion sequences* is defined

$$\mathbf{I}_n^{(\boldsymbol{s})} := \{\boldsymbol{e} \in \mathbb{Z}_{\geq 1}^n \ : \ 0 \leq e_i < s_i\}.$$

The *ascent set* of an inversion sequence $\boldsymbol{e}$ is

$$\mathrm{Asc}(\boldsymbol{e}) = \left\{ i \in \{0, 1, \ldots, n-1\} \ : \ \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\}$$

where $s_0 = 1$ and $e_0 = 0$ by convention, and $\mathrm{asc}(\boldsymbol{e}) = |\,\mathrm{Asc}(\boldsymbol{e})|$.

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Example

Let $\boldsymbol{s} = (2, 5, 7)$. Then $\boldsymbol{e} = (1, 1, 4) \in \mathbf{I}_3^{(2,5,7)}$, $\operatorname{Asc}(\boldsymbol{e}) = \{0, 2\}$ and $\operatorname{asc}(\boldsymbol{e}) = 2$.

Definition

Given $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$, the *s-lecture hall simplex* is the lattice simplex given by

$$\mathbf{P}_n^{(\boldsymbol{s})} := \left\{ \lambda \in \mathbb{R}^n \, : \, 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}$$

or alternatively

$$\mathbf{P}_n^{(\boldsymbol{s})} = \mathrm{conv}\{(0, \ldots, 0, 0), (0, \ldots, 0, s_i, s_{i+1}, \ldots, s_{n-1}, s_n)\}$$

for all $1 \leq i \leq n$

Lecture hall simplex properties

- $h^*(\mathbf{P}_n^{(\boldsymbol{s})}, z) = \sum_{\boldsymbol{e} \in \mathbf{I}_n^{(\boldsymbol{s})}} z^{\mathrm{asc}(\boldsymbol{e})}$ called the $\boldsymbol{s}$-*Eulerian polynomials* (Savage-Schuster [6]). Moreover, these polynomials are real-rooted and hence unimodal (Savage-Visontai [7]).

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- $\mathbf{P}_n^{(\boldsymbol{s})}$ have the integer-decomposition property (monotone $\boldsymbol{s}$ Hibi-O.-Tsuchiya [3], generality Brändén-Solus [2])
- Some partial reflexive/Gorenstein results (Hibi-O.-Tsuchiya [3]).

Gorenstein and level algebras– Abstractly

Let $\mathcal{R} = \bigoplus_{i \in \mathbb{Z}} \mathcal{R}_i$ be a finitely generated $\mathbb{Z}$-graded, d-dimensional k-algebra. Suppose that $\mathcal{R}$ is local and Cohen-Macaulay.

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We say that $\mathcal{R}$ is *level* if the canonical module $\omega_{\mathcal{R}}$ is generated by elements of the same degree, that is $\omega_{\mathcal{R}}$ has minimal generating set $\{\sigma_1, \ldots, \sigma_j\}$ such that $\deg(\sigma_1) = \deg(\sigma_2) = \cdots = \deg(\sigma_j)$.

Equivalently, $\mathcal{R}$ is level if for any h.s.o.p. $\theta_1, \ldots, \theta_d$ of $\mathcal{R}$, all the elements of $\mathrm{soc}(\mathcal{R}/(\theta_1, \ldots, \theta_d))$ are of the same degree.

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(M an $\mathcal{R}$-module

$$\mathrm{soc}(M) := \{u \in M \,:\, R_+ u = 0\}$$

where R_+ is the unique maximal ideal of $\mathcal{R}$)

Gorenstein and Level algebras– Polytopes

For a lattice polytope $\mathcal{P} \subset \mathbb{R}^n$, $k[\mathcal{P}] = k[\boldsymbol{x}^\alpha \cdot t] \subset k[x_1^\pm, \ldots, x_n^\pm, t]$ for all $\boldsymbol{x}^\alpha \cdot t \in \mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}$ where $\mathrm{cone}(\mathcal{P}) = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{(p, 1) : p \in \mathcal{P}\} \subset \mathbb{R}^{n+1}$.

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Gorenstein:

$k[\mathcal{P}]$ is Gorenstein $\Leftrightarrow$

- There exists some $c \in \mathbb{R}^{n+1}$ such that $c + (\mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}) = \mathrm{cone}(\mathcal{P})^\circ \cap \mathbb{Z}^{n+1}$.

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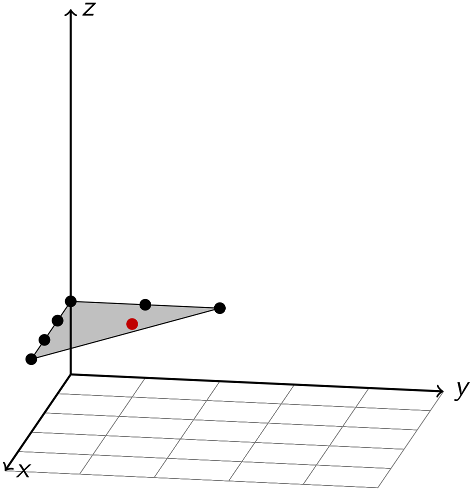
Level:

$k[\mathcal{P}]$ is level $\Leftrightarrow$ there is a finite collection $c_1, \ldots, c_j \in \mathbb{Z}^{n+1}$ where

$$\sum_{i=1}^{j} c_i + (\mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^{n+1}) = \mathrm{cone}(\mathcal{P})^\circ \cap \mathbb{Z}^{n+1}$$

such that all $c_{1_{n+1}} = \cdots c_{j_{n+1}}$.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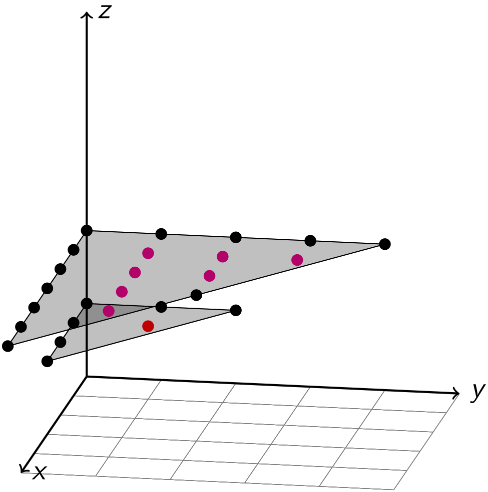
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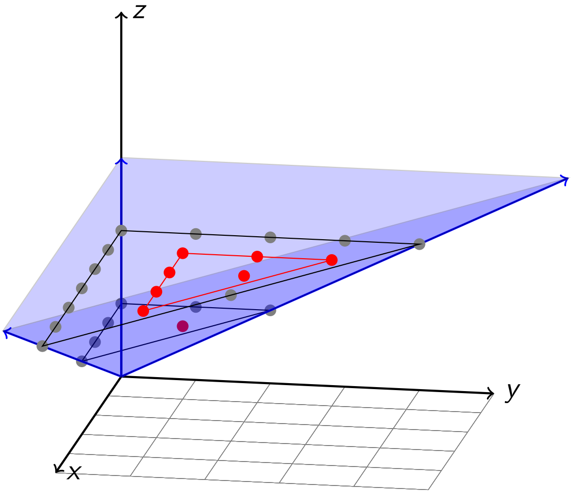
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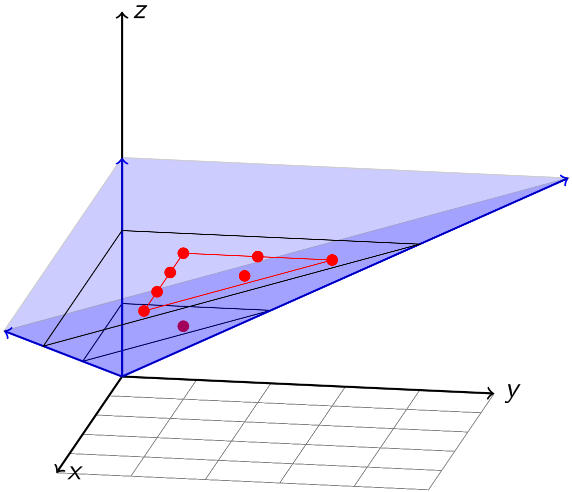

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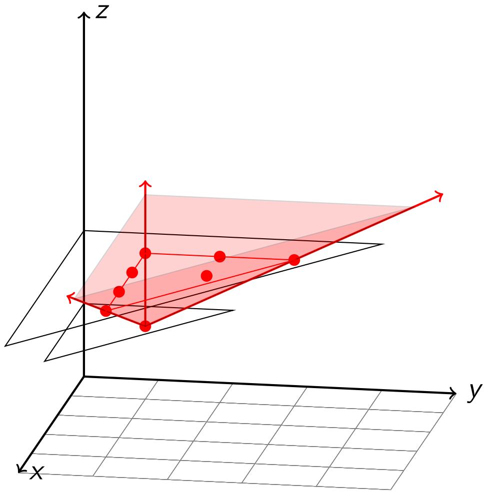
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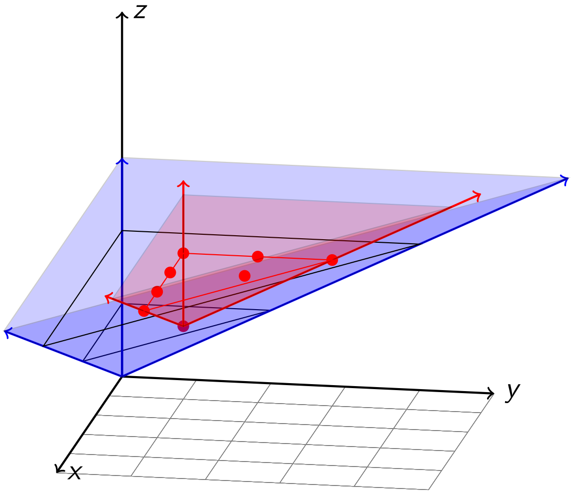
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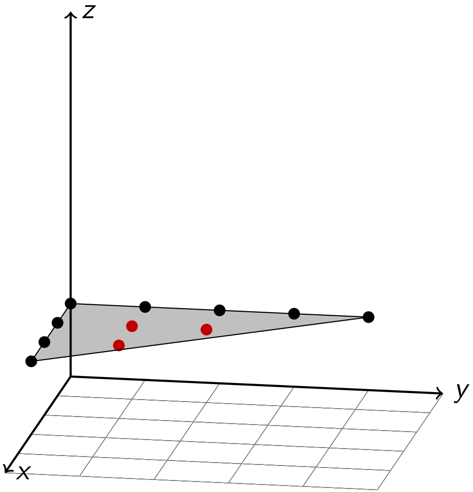
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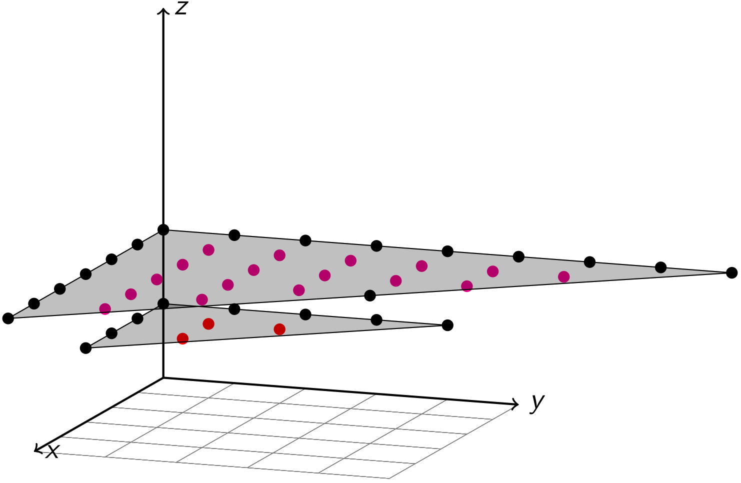
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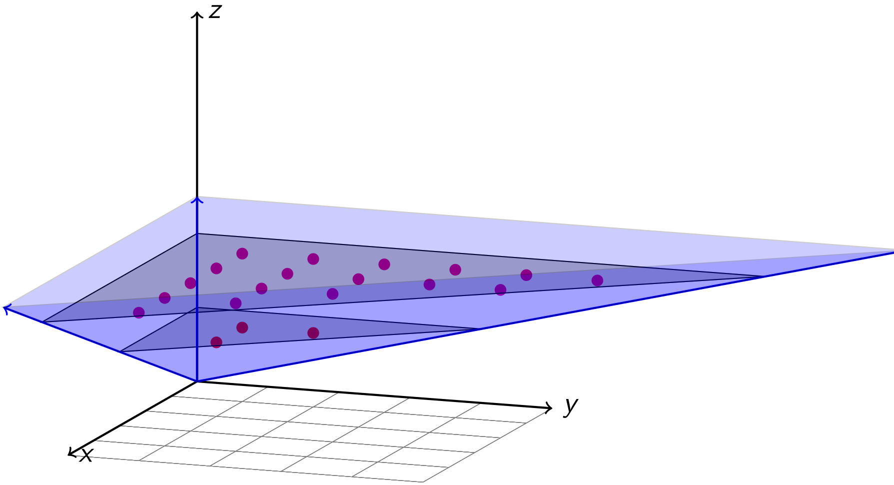
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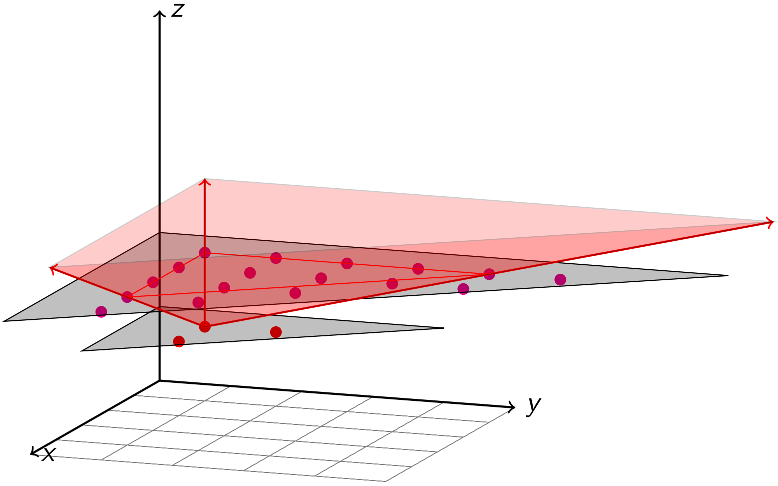

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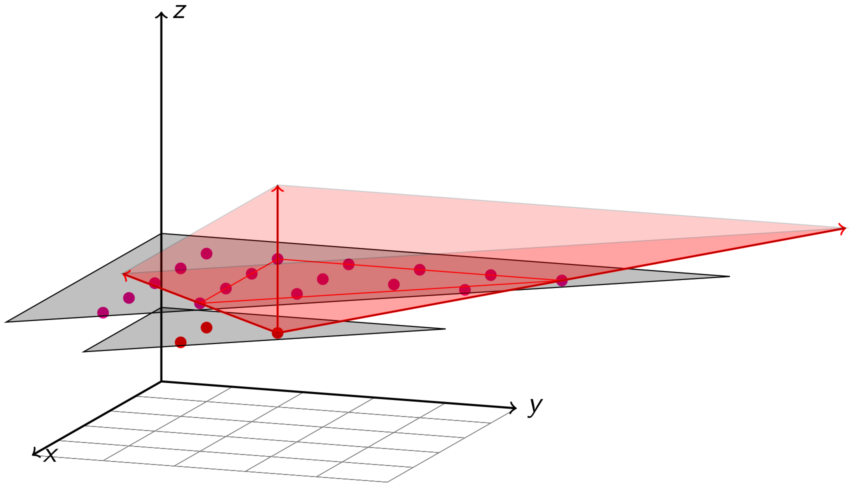
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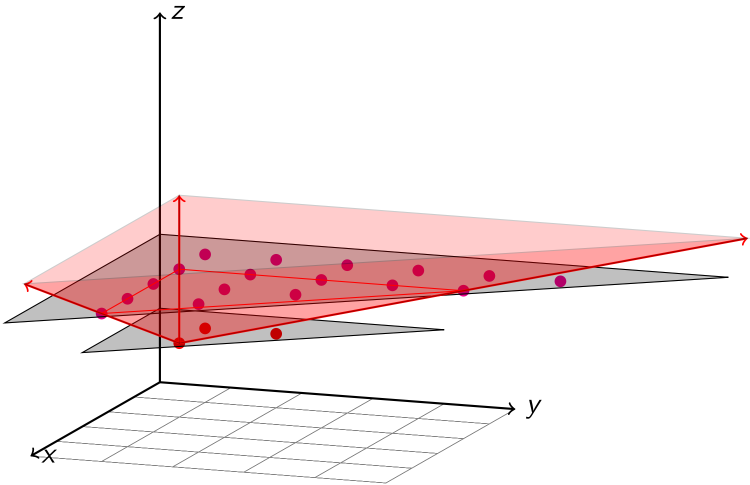
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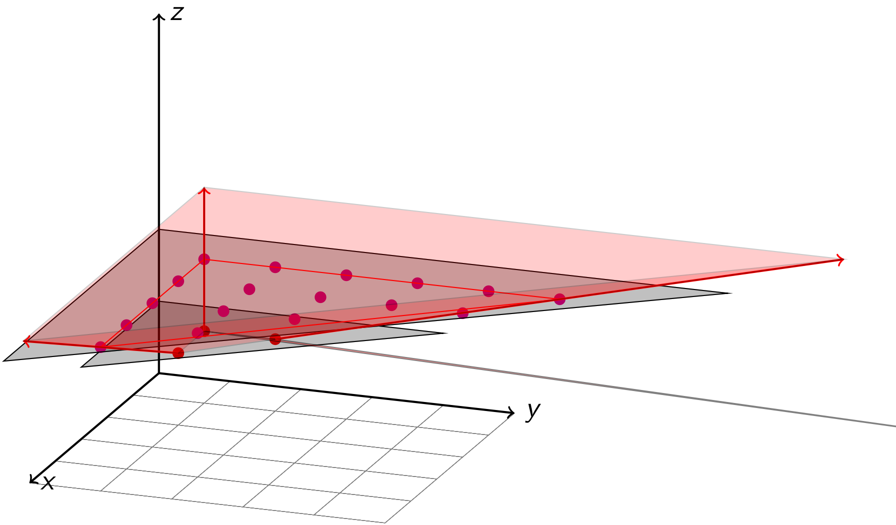
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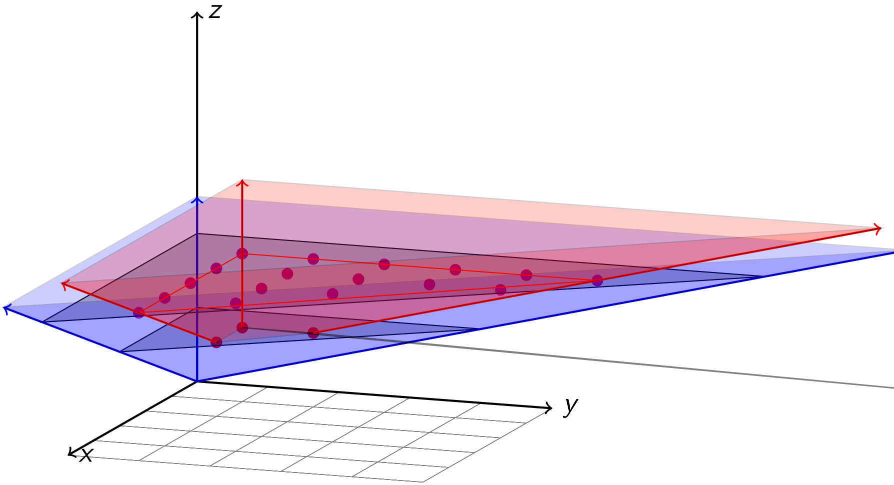
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Recalling inversion sequences and statistics

Recall:

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We define $\mathbf{I}_{n,k}^{(\boldsymbol{s})} = \{\boldsymbol{e} \in \mathbf{I}_n^{(\boldsymbol{s})} \,:\, \mathrm{asc}(\boldsymbol{e}) = k\}$.

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Given $\mathbf{e}, \mathbf{e}' \in \mathbf{I}_n^{(\mathbf{s})}$ s.t. $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ and $\mathbf{e}' = (e_1', e_2', \ldots, e_n')$, then $\mathbf{e} + \mathbf{e}' = (e_1 + e_1' \bmod s_1, e_2 + e_2' \bmod s_2, \ldots, e_n + e_n' \bmod s_n)$.

Theorem (Kohl-O., 2017+, [5])

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ *and let* $r = \max\{\mathrm{asc}(\boldsymbol{e}) : \boldsymbol{e} \in \mathbf{I}_n^{(\boldsymbol{s})}\}$. *Then* $\mathbf{P}_n^{(\boldsymbol{s})}$ *is level if and only if for any* $\boldsymbol{e} \in \mathbf{I}_{n,k}^{(\boldsymbol{s})}$ *with* $1 \leq k < r$ *there exists some* $\boldsymbol{e}' \in \mathbf{I}_{n,1}^{(\boldsymbol{s})}$ *such that* $(\boldsymbol{e} + \boldsymbol{e}') \in \mathbf{I}_{n,k+1}^{(\boldsymbol{s})}$.

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Proof Idea:
Consider $k[\mathbf{P}_n^{(\mathbf{s})}]/(v_0, v_1, \ldots, v_n)$ where $v_0 = (0, \ldots, 0, 1)$ and $v_i = (0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n)$ for all i.

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The condition on inversion sequences in the theorem is now precisely the necessary condition for $\mathrm{soc}(k[\mathbf{P}_n^{(\boldsymbol{s})}]/(v_0, v_1, \ldots, v_n))$ to contain elements of a single degree.

Consequences

Corollary

For any $\boldsymbol{s} = (s_1, s_2)$, $\mathbf{P}_2^{(s_1,s_2)}$ *is level.***

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Suppose that $\boldsymbol{s} = (s_1, \ldots, s_d)$ *and* $\boldsymbol{t} = (t_1, \ldots, t_e)$ *are sequence such that* $\mathbf{P}_d^{(\boldsymbol{s})}$ *and* $\mathbf{P}_e^{(\boldsymbol{t})}$ *are level polytopes. Then*

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- $\mathbf{P}_{d+1}^{(1,\boldsymbol{s})}$ *is level;*

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- $\mathbf{P}_{e+1}^{(\boldsymbol{t},1)}$ *is level;*

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- $\mathbf{P}_{d+e+1}^{(\boldsymbol{s},1,\boldsymbol{t})}$ *is level*

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For any $\boldsymbol{s} = (s_1, s_2)$, $\mathbf{P}_2^{(s_1,s_2)}$ *is level.***

**It is already known that all lattice polygons are level (Higashitani-Yanagawa [4]), but one can use the theorem to proves this.

Corollary

Suppose that $\boldsymbol{s} = (s_1, \ldots, s_d)$ *and* $\boldsymbol{t} = (t_1, \ldots, t_e)$ *are sequence such that* $\mathbf{P}_d^{(\boldsymbol{s})}$ *and* $\mathbf{P}_e^{(\boldsymbol{t})}$ *are level polytopes. Then*

- $\mathbf{P}_{d+1}^{(1,\boldsymbol{s})}$ *is level;*
- $\mathbf{P}_{e+1}^{(\boldsymbol{t},1)}$ *is level;*
- $\mathbf{P}_{d+e+1}^{(\boldsymbol{s},1,\boldsymbol{t})}$ *is level*

We can use these to create an infinite family of lecture hall simplies which are level of arbitrarily high dimension.

Overview

Theorem (Kohl-O., 2017+, [5])

Let $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n_{\geq 1}$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein if and only if there exists a $\mathbf{c} \in \mathbb{Z}^{n+1}$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$ and

$$c_{n+1} s_n = 1 + c_n$$

with $c_1 = 1$.

Theorem (Kohl-O., 2017+, [5])

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ be a sequence such that there exists some $1 < i \leq n$ such that $\gcd(s_{i-1}, s_i) = 1$ and define $\overleftarrow{\boldsymbol{s}} = (\overleftarrow{s_1}, \ldots, \overleftarrow{s_n}) := (s_n, s_{n-1}, \ldots, s_1)$. Then $\mathbf{P}_n^{(\boldsymbol{s})}$ is Gorenstein if and only if there exists $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

and

$$d_j \overleftarrow{s_{j-1}} = d_{j-1} \overleftarrow{s_j} + \gcd(\overleftarrow{s_{j-1}}, \overleftarrow{s_j})$$

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Gorenstein lecture hall polytopes

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for $j > 1$ *with* $c_1 = d_1 = 1$.

Proof uses ideas from Beck-Braun-Köppe-Savage-Zafeirakopoulos [1] result on Gorenstein lecture hall cones.

Some remaining questions

Conjecture (Kohl-O.)

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n_{\geq 1}$. *Suppose that there exists a* $\boldsymbol{c} \in \mathbb{Z}^{n+1}$ *satisfying*

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$. *Then* $\mathbf{P}_n^{(\boldsymbol{s})}$ *is level.*

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Questions:

- If $\mathcal{P}$ is a polytope with the IDP such that at least one vertex cone of $\mathcal{P}$ is a Gorenstein cone, is $\mathcal{P}$ a level polytope?

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- If $\mathcal{P}$ is a polytope with the IDP such that at least one vertex cone of $\mathcal{P}$ is a Gorenstein cone, is $\mathcal{P}$ a level polytope?
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- If $\mathcal{P}$ is a polytope with the IDP such that at least one vertex cone of $\mathcal{P}$ is a Gorenstein cone, is $\mathcal{P}$ a level polytope?
- Will the levelness proof method adapt nicely to other families of lattice polytopes? (i.e. can we do something similar to other simplies with "combinatorially nice" parallelpipeds?)
- Are there reasonably nice conditions for when $\mathcal{P} \oplus \mathcal{Q}$ is a level polytope?

Overview

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Thank you!

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