K3 polytopes and their quartic surfaces

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Tropical hypersurfaces

K3 polytopes

Stablity of quartic surfaces

Tropical arithmetic

We work over the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$.
Tropical operations are defined as follows:

$$a \oplus b = \min\{a, b\} \text{ and } a \odot b = a + b.$$

They are associative and commutative.

Identity elements: $a \oplus \infty = a$ and $a \odot 0 = a$.

Multiplication is distributive with respect to addition.

Tropical polynomials

Let $x_1, x_2, \ldots, x_n$ be variables representing elements in the tropical semiring. A monomial is any product of variables:

$$x_1^{i_1} \odot x_2^{i_2} \odot \cdots \odot x_n^{i_n}.$$

A polynomial is a finite linear combinations of monomials:

$$f(x_1, x_2, \ldots, x_n) = a_i \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus a_j \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots .$$

We allow negative exponents.

Remark

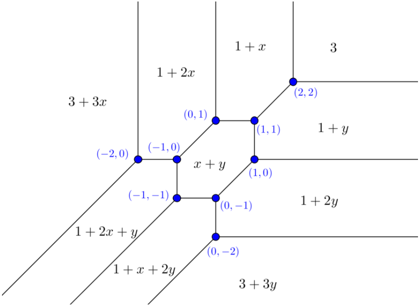
Polynomials "translate" to piecewise-linear concave functions:

$$f(x) = \min\{a_i + i_1 x_1 + i_2 x_2 + \cdots + i_n x_n,\ a_j + j_1 x_1 + j_2 x_2 \cdots j_n x_n,\ \cdots\}.$$

Tropical hypersurfaces

Given a polynomial f, we define the hypersurface $T(f)$ of f as the set of points $x \in \mathbb{R}^n$ at which *the minimum is attained at least twice*.

Example:
$f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$



Tropical hypersurfaces

Theorem (Structure theorem)

The tropical hypersurface $T(f)$ is the support of a pure rational polyhedral complex of dimension $n-1$.

The closure of the connected components of the complement of a tropical hypersurface $T(f)$ are called regions of $T(f)$. They are convex polyhedra.

Tropical and classical hypersurfaces

Let K be an algebraically closed field with nontrivial valuation (for example $K = \mathbb{C}\{\{t\}\}$). Let f be an Laurent polynomial

$$f = \sum_{u=(u_1,\dots,u_n)\in\mathbb{Z}^n} c_u x_1^{u_1}\cdots x_n^{u_n}, \text{with } c_u \in K.$$

We define its tropicalization $\mathrm{trop}(f)$ as

$$\mathrm{trop}(f) = \min_{u\in\mathbb{Z}^n}\Big\{\mathrm{val}(c_u) + \sum_{i\leq n} u_i\, x_i\Big\}.$$

Theorem (Kapranov's theorem)

The following sets coincide:

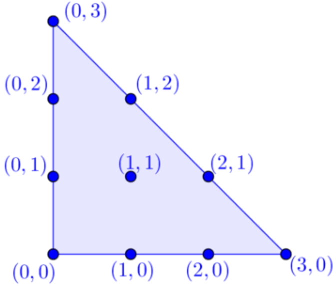
1. $\{w \in \mathbb{R}^n \,\big|\,$ *the min in* $trop(f)(w)$ *is attained at least twice*$\}$;
2. *the closure of* $\{(val(y_1), \dots, val(y_n)) \,|\, (y_1, \dots, y_n) \in V(f)\}$.

Newton polytopes

Given a tropical polynomial $f = \bigoplus_{v \in \mathbb{Z}^n} a_v x^v$, we define the Newton polytope $\mathrm{Newt}(f)$ as the polytope

$$\mathrm{Newt}(f) = \mathrm{conv}(v \, : \, a_v \neq \infty).$$

Example: $f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$



Newton polytopes

We consider the convex hull in $\mathbb{R}^{n+1}$ of the points (v, a_v). The projection of the lower faces on $\mathrm{Newt}(f)$ induces a subdivision of the Newton polytope.

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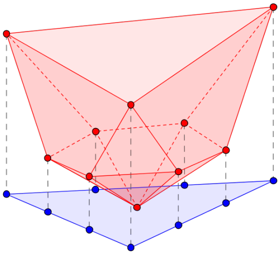

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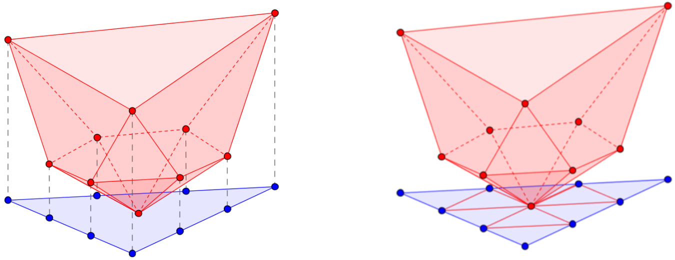

Newton polytopes and tropical hypersurfaces

Tropical hypersurfaces are dual to the regular subdivision of their Newton polytopes induced by the coefficients.

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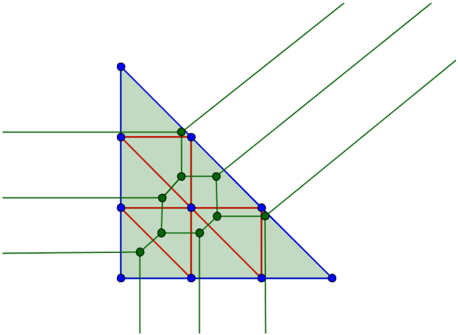
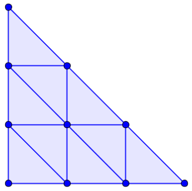


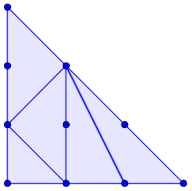
Smooth tropical hypersurfaces

A tropical hypersurface is smooth if the regular subdivision induced by its coefficients is a unimodular triangulation, i.e., cells in the subdivision are simplices of minimal volume $\frac{1}{n!}$.

Examples:



Unimodular



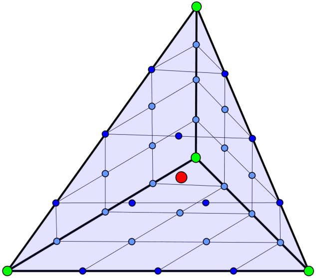
Not unimodular

Regions of a tropical quartic surface

Let $4\Delta_3$ be the 4-th dilatation of the standard simplex,

$$4\Delta_3 = \text{conv}\Big((0,0,0),(4,0,0),(0,4,0),(0,0,4)\Big)$$

The point $\mathbf{p} = (1,1,1)$ is the unique interior lattice point of $4\Delta_3$.



Let $T(f)$ be a smooth tropical hypersurface of degree 4 in $\mathbb{R}^3$. Its Newton polytope $\text{Newt}(f)$ is contained in $4\Delta_3$. If $\text{Newt}(f)$ contains $\mathbf{p}$ in its relative interior, the hypersurface cuts one bounded region out.

Definition

A 3-dimensional polytope $\mathcal{P}$ is a K3 polytope if it arises as the closure of the bounded region in the complement of a smooth tropical surface of degree 4.

Example

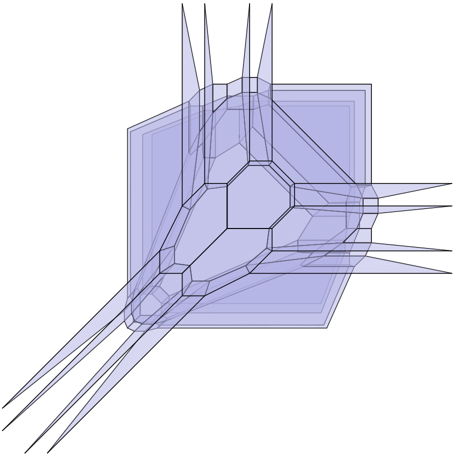
Consider the tropical quartic surface defined by the polynomial:

$$
\begin{aligned}
f = 5(x^4 \oplus y^4 \oplus z^4) \oplus 3(x^3y \oplus x^3z \oplus xy^3 \oplus y^3z \oplus xz^3 \oplus yz^3) \\
\oplus 2(x^2y^2 \oplus x^2z^2 \oplus y^2z^2) \oplus 0(x^2yz \oplus xy^2z \oplus xyz^2) \oplus 3(x^3 \oplus y^3 \oplus z^3) \\
\oplus 0(x^2y \oplus x^2z \oplus xy^2 \oplus y^2z \oplus xz^2 \oplus yz^2) \oplus 2(x^2 \oplus y^2 \oplus z^2) \\
\oplus 0(xy \oplus xz \oplus yz) \oplus 3(x \oplus y \oplus z) \oplus (-9xyz) \oplus 5.
\end{aligned}
$$

The Newton polytope $\mathrm{Newt}(f)$ is $4\Delta_3$.

Let's look at the K3 polytope defined by $T(f)$

This is the smooth tropical quartic surface $T(f)$:



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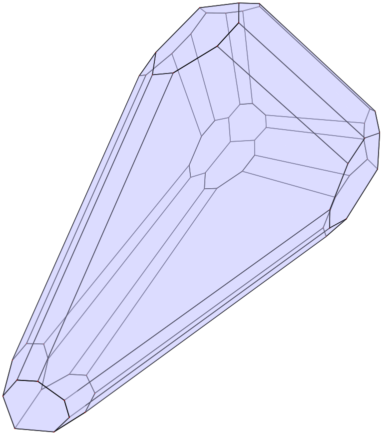
This is the K3 polytope:



Its f-vector is $(64, 96, 34)$.

The hunt for K3 polytopes

Smooth tropical quartic surfaces are dual to regular unimodular triangulations of their Newton polytopes. We switch our attention to these objects.

From now on we will only talk about Newton polytopes P contained in $4\Delta_3$ containing $\mathbf{p} = (1, 1, 1)$ in their relative interior. They are canonical polytopes.

The hunt for K3 polytopes

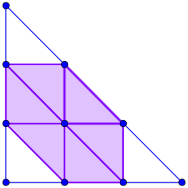
Warning: computing the secondary fan of $4\Delta_3$ does not seem feasible.

- $2\Delta_3$ has 10 lattice points and 15 regular triangulations,
- $3\Delta_3$ has 20 lattice points and 21 125 102 regular triangulations.

Triangulations of $3\Delta_3$ were computed by Jordan, Joswig and Kastner with `MPTOPCOM`.

Central triangulations

We define the central part of a triangulation $\mathcal{T}$ as the subset of $\mathcal{T}$ given by the simplices of $\mathcal{T}$ containing $\mathbf{p}$. If $\mathcal{T}$ coincides with its central part, then we say that $\mathcal{T}$ is central.



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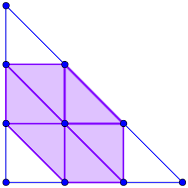

The K3 polytope is uniquely determined by the central part of the triangulation $\mathcal{T}$.

It is enough to consider central triangulations!

Central traingulations of canonical polytopes

How can we construct central triangulations of a canonical polytope?

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A polytope P is reflexive if $A\mathbf{p} - c = 1$, where $Ax \geq c$ are the equations defining P.
If P is reflexive,

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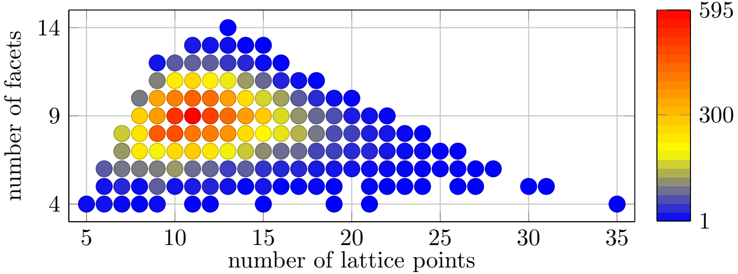
A three dimensional canonical lattice polytope P is reflexive if and only if every central fine triangulation of P is unimodular. We need to consider reflexive polytopes!

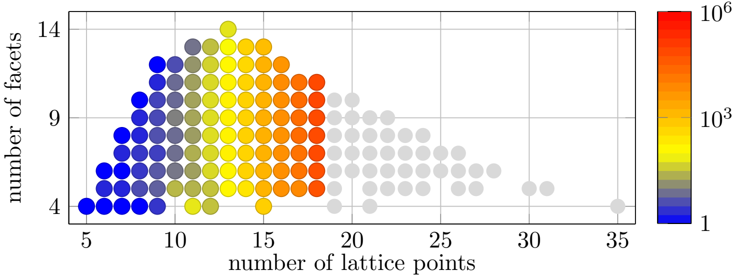
Theorem (Balletti-P-Sturmfels)

Up to symmetry there are 15139 possible reflexive polytopes contained in $4\Delta_3$.



Theorem (Balletti-P-Sturmfels)

The reflexive polytopes of volume ≤ 30 *in the theorem above admit a total of* 36 297 333 *regular unimodular central triangulations. Every K3 polytope with* ≤ 30 *vertices arises from one of these.*



f-vectors of K3 polytopes

Theorem (Balletti-P-Sturmfels)

Let $\mathcal{P}$ be a K3 polytope obtained from a polytope P. Then $\mathcal{P}$ is a simple polytope. The entries of its f-vector are

$$v = vol(P), \quad e = \frac{3vol(P)}{2}, \quad f = |P \cap \mathbb{Z}^2| - 1.$$

f-vect	# of P	f-vect	# of P	f-vect	# of P
(4, 6, 4)	9	(22, 33, 13)	1248	(40, 60, 22)	27
(6, 9, 5)	102	(24, 36, 14)	922	(42, 63, 23)	18
(8, 12, 6)	412	(26, 39, 15)	628	(44, 66, 24)	7
(10, 15, 7)	959	(28, 42, 16)	465	(46, 69, 25)	9
(12, 18, 8)	1642	(30, 45, 17)	295	(48, 72, 26)	2
(14, 21, 9)	2083	(32, 48, 18)	203	(50, 75, 27)	2
(16, 24, 10)	2194	(34, 51, 19)	128	(54, 81, 29)	1
(18, 27, 11)	1997	(36, 54, 20)	85	(56, 84, 30)	1
(20, 30, 12)	1646	(38, 57, 21)	53	(64, 96, 34)	1

Moduli space of quartic surfaces

A quartic surface is the variety in $\mathbb{P}^3$ defined by a homogeneous polynomial of degree 4 in $\mathbb{C}[x, y, z, w]$,

$$f(x, y, z, w) \quad = \sum_{i+j+k \leq 4} c_{ijk} x^i y^j z^k w^{4-i-j-k}.$$

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The 35 coefficients c_{ijk} parameterize all the quartic surfaces. So we consider the 34-dimesional projective space $\mathbb{HS}_{4,3} = \mathbb{P}(\mathrm{HS}_{4,3})$ of quartic surfaces. This gives us a "moduli space of quartic surfaces".

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More precisely, the special linear group SL(4) acts on $\mathbb{HS}_{4,3}$, and on the associated polynomial ring $\mathbb{C}[\mathbb{HS}_{4,3}]$, generated by c_{ijk}. The moduli space of quartic surfaces in $\mathbb{P}^3$ is the projective variety determined by $\mathrm{Proj}\big(\mathbb{C}[\mathbb{HS}_{4,3}]^{\mathsf{SL}(4)}\big)$.

Stable elements

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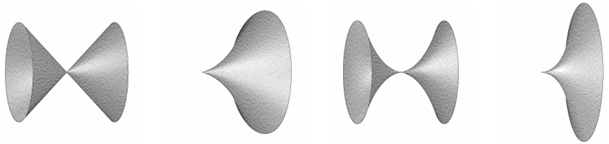
Determining whether an element is stable is connected to the study of its singular locus.

Theorem (Shah '81)

If the singular locus of a quartic surface contains at most rational double points, then the surface is stable.

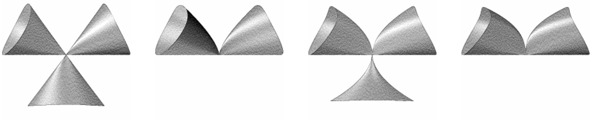
Arnold's classification '72

A_k: $x^2 + y^2 + z^{k+1}$



A_1 A_2 A_3 A_4

D_k: $x^2 + y^2 z + z^{k-1}$



D_4 D_5 D_6 D_7

Pictures from Greuel-Lossen-Shustin "Introduction to Singularities and Deformation".

Arnold's classification '72

E_6: $x^2 + y^3 + z^4$, E_7: $x^2 + y^3 + yz^3$, and E_8: $x^2 + y^3 + z^5$

E_6

E_7

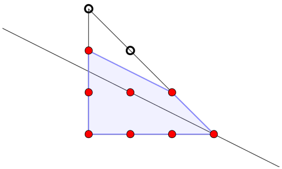
E_8

Pictures from Greuel-Lossen-Shustin "Introduction to Singularities and Deformation".

A combinatorial criterion

Theorem (Mumford '77)

A point f in $\mathbb{HS}_{4,3}$ is stable if and only if, for every choice of coordinates, and for all planes H through $\mathbf{p}$, each open halfspace of H contains a monomial of f.



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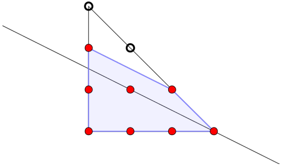

A reflexive lattice polytope P contained in $4\Delta_3$ is called minimal if it does not properly contain any reflexive polytope.

There are precisely 115 minimal reflexive polytopes in $4\Delta_3$.

Stability

Theorem (Balletti-P-Sturmfels)

Let $f \in \mathbb{C}[x, y, z, w]$ be a generic homogeneous quartic surface whose Newton polytope arises from a smooth tropical surface. Then the quartic surface $V(f)$ in $\mathbb{P}^3$ is stable.

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- We show the stability of surfaces having a minimal polytope as Newton polytope by studying their singular locus.
- We use Mumford's criterion to conclude that also generic surfaces with Newton polytope containing a minimal one are stable.

Thank you!