A McMullen's formula for Ehrhart coefficients

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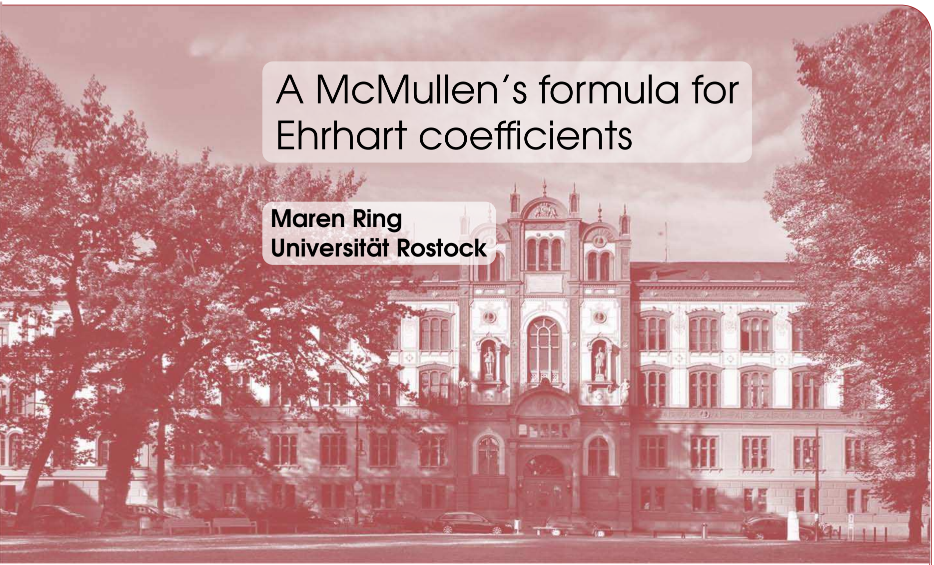
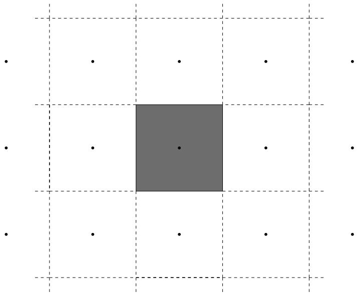


Lattice



Let V be a Euclidean space and $L \subset V$ be a lattice.

A *fundamental domain* T of L is a bounded subset $T \subseteq V$, such that

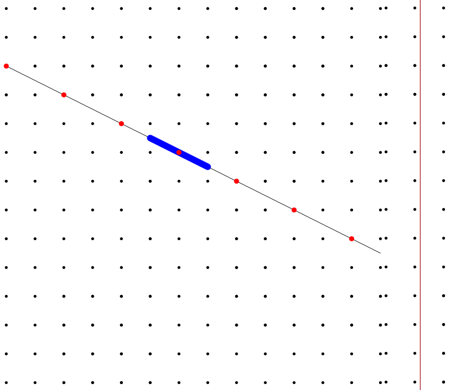
- $\bigcup_{x \in L} (T + x) = \operatorname{span}(L)$,
- $(T + x) \cap (T + y) = \varnothing$ for $x, y \in L$ with $x \neq y$ and
- every intersection of T with an affine subspace is Lebesgue measurable.

The relative volume

The *relative volume* $\mathrm{vol}_L(A)$ of a set A with respect to a lattice L is given by

$$\mathrm{vol}_L(A) = \frac{\mathrm{absvol}_{\mathrm{span}(L)}(A)}{\mathrm{absvol}_{\mathrm{span}(L)}(T)}$$

for any fundamental domain T of L.



The Ehrhart polynomial

Let L be a lattice and P a d-dimensionale lattice polytope. Then

$$Ehr_P(k) := |kP \cap L| = e_d k^d + e_{d-1} k^{d-1} + \ldots + e_1 k + e_0$$

(Ehrhart 1962)

Coefficients?

$$e_d = \text{vol}(P), \quad e_{d-1} = \frac{1}{2} \sum_{f \in F_{d-1}} \text{vol}(f), \quad e_0 = 1$$

2-dimensional:

$$Ehr_P(k) = |kP \cap \mathbb{Z}^2| = \text{vol}(P) k^2 + \frac{1}{2} \sum_{f \in F_1} \text{vol}(f) k + 1$$

(Pick 1899)

The Ehrhart polynomial

$$Ehr_P(k) = e_d k^d + e_{d-1} k^{d-1} + \ldots + e_1 k + e_0$$

(Ehrhart 1962)

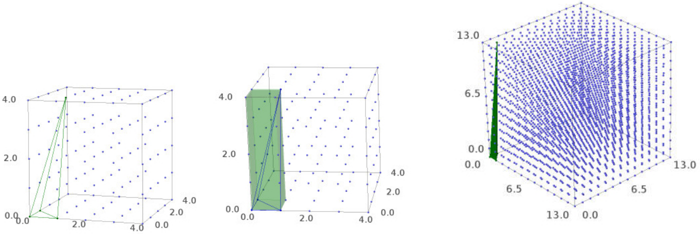
Coefficients?

All positive?

Sadly no...

Reeve tetrahedra

$$T_N = \mathrm{conv}((0,0,0),(1,0,0),(0,1,0),(1,1,N))$$



$$4 = Ehr_{T_N}(1) = \mathrm{vol}(T_N) + \frac{1}{2}\sum_{f\in F_2}\mathrm{vol}(f) + e_1^{(N)} + 1$$

The Ehrhart polynomial

$$Ehr_P(k) = e_d k^d + e_{d-1} k^{d-1} + \ldots + e_1 k + e_0$$

(Ehrhart 1962)

Coefficients?

$$e_d = \text{vol}(P), \quad e_{d-1} = \frac{1}{2} \sum_{f \in F_{d-1}} \text{vol}(f), \quad e_0 = 1$$

Generalizations?

McMullen's formulas!

McMullen´s formulas

aka local fromulas for Ehrhart coefficients

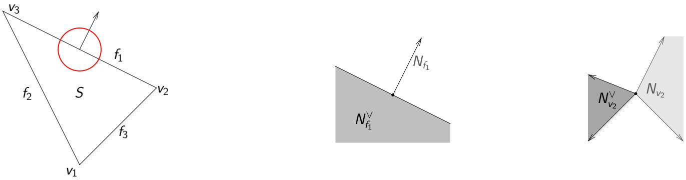
Functions μ exist, that assign a (rational) number to every cone, such that the following equalities for the Ehrhart coefficients hold:

$$e_i = \sum_{f \in F_i} \mu(N_f)\, \mathrm{vol}(f) \qquad \text{for all } i \in \{1, \ldots, d\}$$

(McMullen1983)

N_f=*normal cone* of the face f, i.e. the cone over the outer normal vectors of the facets meeting in f

Locality



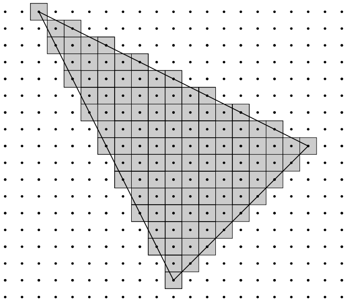
Known Constructions

Robert Morelli (1993), *Pick's theorem and the Todd class of a toric variety*, Adv. Math. 100, 183-231.

James E. Pommersheim and Hugh Thomas (2004), *Cycles representing the Todd class of a toric variety*, J. Amer. Math. Soc. 17, 983-994.

Nicole Berline and Michèle Vergne (2007), *Local Euler McLaurin formula for polytopes*, Mosc. Math. J., Volume 7, Number 3, 355–386.

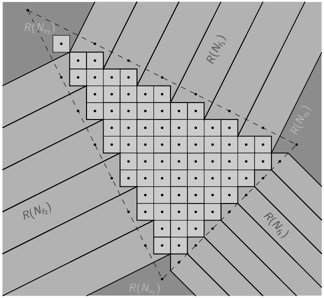
Idea



$$|P \cap L| = \mathrm{vol}\underbrace{\left(\bigcup_{x \in |P \cap L|} (x + T)\right)}_{:= \textit{cellcomplex} \text{ of P}},$$

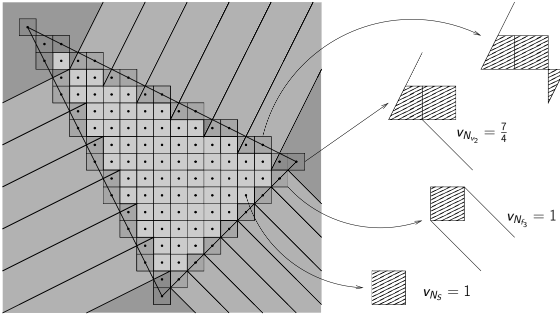
for any fundamental domain T of L.

Construction



Define a tiling of space into regions corresponding to normal cones, which are periodic w.r.t. the sublattice in the affine hull of the face.

Construction

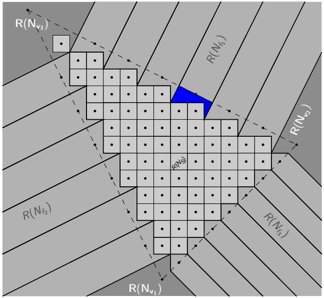


Compute the volume of the cellcomplex inside each region.

$$\mu(N_S) = 1, \quad \mu(N_{f_1}) = 2 -$$

$$\mu(N_{v_2}) = \frac{7}{4} -$$

Construction



An error is made near the boundary when taking

$$e_d = \mu(N_S)\,\mathrm{vol}(S) = 1 \cdot \mathrm{vol}(S)$$

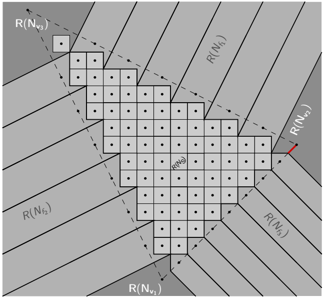
$$e_i = \sum_{f \in F_i} \mu(N_f)\,\mathrm{vol}(f)$$

Substract correction volumes for faces of the normal cone.

$$\mu(N_S) = 1, \quad \mu(N_{f_1}) = 2 - \tfrac{3}{2}\mu(N_S),$$

$$\mu(N_{v_2}) = \tfrac{7}{4} - \tfrac{7}{8}\mu(N_S) -$$

Construction



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$$\mu(N_{v_2}) = \tfrac{7}{4} - \tfrac{7}{8}\mu(N_S) - \; - \tfrac{1}{2}\mu(N_{f_1}) - \tfrac{1}{2}\mu(N_{f_3})$$

formula

After constructing the regions for each cone, we can compute inductively

$$\mu(\{0\}) := 1 \tag{1}$$

and

$$\mu(C) := v_C - \sum_{K<C} w_K^C \cdot \mu(K), \tag{2}$$

for pointed rational cones $C \subseteq V$ with $\dim(C) \geq 1$, where v_C is the volume of the cellcomplex intersected with the region and w_C^K is the correction term.

Symmetry

Let $\mathcal{G}$ be a subgroup of the lattice preserving symmetries of P.

Define a $\mathcal{G}$-invariant scalar product:

$$\langle x, y \rangle_{\mathcal{G}} := x^t G y$$

for all $x, y \in \mathbb{R}^d$, given by the Gram matrix

$$G := \frac{1}{|\mathcal{G}|} \sum_{A \in \mathcal{G}} A^t A$$

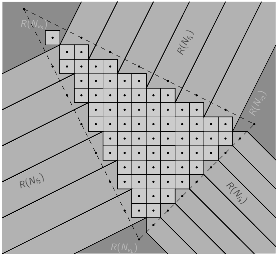
Symmetry

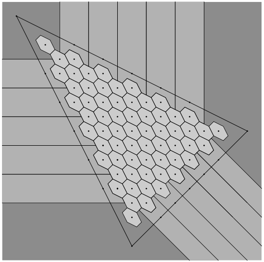
For a given scalar product $\langle \cdot, \cdot \rangle$ and lattice L, the *Dirichlet-Voronoi cell* is defined as

$$DV(\langle , \rangle, L) = \{x \in \mathrm{span}(L) | \langle x, x \rangle \leq \langle x - p, x - p \rangle \ \forall p \in L\}.$$

Trick: An invariant scalar product defines an invariant Dirichlet-Voronoi cell and thus equal values $\mu(N_f)$ for all faces f in the same orbit w.r.t. $\mathcal{G}$.

Symmetry





Face f	S	f_1	f_2	f_3	v_1	v_2	v_3
$\mu(N_f)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$

$\dim(f)$	2	1	0
$\mu(N_f)$	1	$\frac{1}{2}$	$\frac{1}{3}$

Thank you for your attention!