Real-rootedness, Unimodality, and Symmetric Decompositions of Polynomials

Liam Solus

KTH Royal Institute of Technology

solus@kth.se

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A Common Endeavour in Combinatorics:

Prove that a generating polynomial

$$p(z) = p_0 + p_1 z + \cdots + p_d z^d \qquad p_k \in \mathbb{Z}_{\geq 0}, k \in \{0, \ldots, d\}$$

is **unimodal**; i.e.

$$p_0 \leq p_1 \leq \cdots \leq p_t \geq \cdots \geq p_{d-1} \geq p_d \qquad \text{for some } t \in \{0, \ldots, d\}.$$

Also: p(z) is **symmetric with respect to** m if $p_k = p_{m-k}$ for all $k \in \{0, \ldots, m\}$.

Two Common Approaches:

(1) Decompose and use symmetry:

- A product of symmetric and unimodal polynomials is symmetric and unimodal.
- A sum of symmetric and unimodal polynomials with similar modes is unimodal.

(2) Factor and use roots:

- If $p(z)$ is **real-rooted** then it is also unimodal.

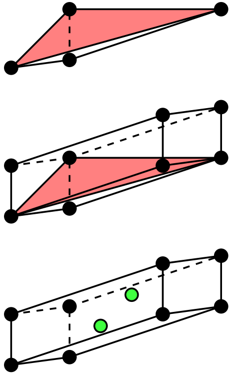
Ehrhart h^*-polynomials:

- $P \subset \mathbb{R}^n$ a lattice polytope of dimension d with **(Ehrhart) h^*-polynomial** $h^*(P;z)$.
- $\Delta = \mathrm{conv}(v^{(0)}, \ldots, v^{(d)}) \subset \mathbb{R}^n$ a lattice d-simplex with **open parallelpiped**

$$\Pi^\circ_\Delta := \left\{ \sum_{i=0}^{d} \lambda_i (v^{(i)}, 1) \in \mathbb{R}^{n+1} \,:\, 0 < \lambda_i < 1 \right\},$$

and **box polynomial** (or **local h^*-polynomial**):

$$\ell^*(\Delta; z) := \sum_{(x_1,\ldots,x_{n+1}) \in \Pi^\circ_\Delta \cap \mathbb{Z}^{n+1}} z^{x_{n+1}}.$$



Theorem (Betke and McMullen; 1985). Let T be a lattice triangulation of a d-dimensional lattice polytope P. Then

$$h^*(P;z) = \sum_{\Delta \in T} h(\mathrm{link}_T(\Delta); z)\,\ell^*(\Delta; z),$$

where $h(\mathrm{link}_T(\Delta); z)$ is the h-polynomial of the link of Δ in T.

A Corollary and the Question of Box Unimodality:

P is **reflexive** if $h^*(P;z)$ is symmetric with respect to d.

Corollary (Betke and McMullen; 1985). Let T be a lattice triangulation of **the boundary** of a d-dimensional reflexive polytope P. Then

$$h^*(P;z) = \sum_{\Delta \in T} h(\mathrm{link}_T(\Delta);z)\ell^*(\Delta;z).$$

- $\ell^*(\Delta;z)$ is always symmetric with respect to $\dim(\Delta)+1$.
- If T is **regular** then $h(\mathrm{link}_T(\Delta);z)$ is symmetric with respect to $d-\dim(\Delta)-1$ and unimodal.

Definition (Schepers and Van Langenhoven; 2013). A lattice triangulation of P is **box unimodal** if it is regular and $\ell^*(\Delta;z)$ is unimodal for all $\Delta \in T$.

- Reflexive polytopes with box unimodal triangulations have unimodal h^*-polynomials.

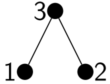
Question (Schepers and Van Langenhoven; 2013). Which (IDP and reflexive) lattice polytopes have box unimodal triangulations?

s-Lecture Hall Order Polytopes:

- $s = (s_1, \ldots, s_n)$ a sequence of positive integers.

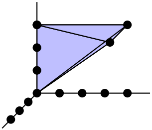
$s = (s_1, s_2, s_3) = (2, 4, 3)$

- $\mathcal{P} := ([n], \preceq)$ a naturally-labeled poset on ground set $[n]$.



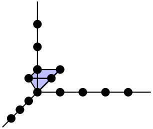
- The s-**lecture hall simplex** is

$$P_n^s := \left\{ x \in \mathbb{R}^n \, : \, 0 \leq \frac{x_1}{s_1} \leq \cdots \leq \frac{x_n}{s_n} \leq 1 \right\}$$



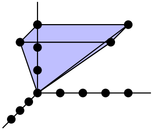
- The **order polytope** is

$$O(\mathcal{P}) := \{ x \in \mathbb{R}^n \, : \, 0 \leq x_i \leq 1, x_i \leq x_j \text{ if } i \preceq j \}$$



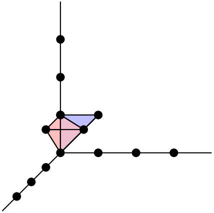
- The s-**lecture hall order polytope** is

$$O(\mathcal{P}, s) := \left\{ x \in \mathbb{R}^n \, : \, 0 \leq x_i \leq s_i, \frac{x_i}{s_i} \leq \frac{x_j}{s_j} \text{ if } i \preceq j \right\}$$

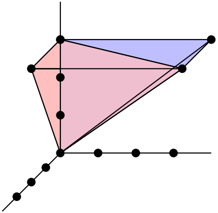


s-Lecture Hall Order Polytopes:

- **(Stanley; 1986).** $\mathcal{O}(\mathcal{P})$ admits a regular triangulation into unimodular simplices, called the **canonical triangulation** of $\mathcal{O}(\mathcal{P})$.



- **(Brändén, Leander; 2016).** $\mathcal{O}(\mathcal{P}, s)$ admits a regular triangulation into s-lecture hall simplices, called the s-**canonical triangulation** of $\mathcal{O}(\mathcal{P}, s)$.



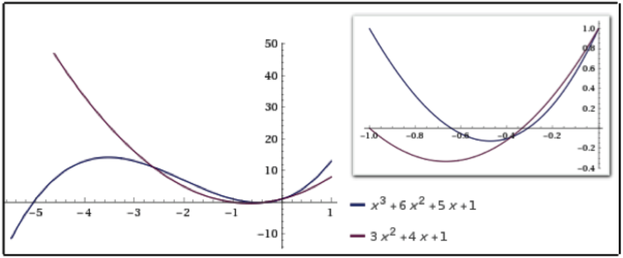
Question. Is the s-canonical triangulation always box unimodal? First of all, is $\ell^*(P_n^s; z)$ always unimodal?

Use the Roots!:

Theorem (Gustafsson, LS; 2018). The local h^*-polynomial $\ell^*(P_n^s; z)$ is always real-rooted, and thus unimodal.

The Proof:

- $\ell^*(P_n^s; z)$ admits a combinatorial interpretation using **descent statistics** of **inversion sequences**.
- Use the theory of **interlacing polynomials**.



If q interlaces p then $p+q$ is real-rooted! (Use recursions!)

Corollary (Gustafsson, LS; 2018). The s-canonical triangulation of $O(\mathcal{P}; s)$ is box unimodal. Moreover, if $O(\mathcal{P}, s)$ is reflexive then $h^*(O(\mathcal{P}, s); z)$ is unimodal.

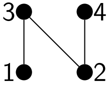
An Application:

Theorem (Brändén, Leander; 2016). Let $\mathcal{P} = ([n], \preceq)$ be naturally-labeled, ranked poset with rank function $\rho : [n] \longrightarrow \mathbb{Z}_{\geq 0}$, and let

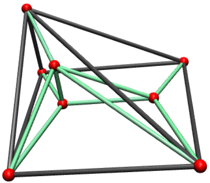
$$s_i := \rho(i) + 1 \qquad \text{for all } i \in [n].$$

Then $h^*(O(\mathcal{P}, s); z)$ is symmetric with respect to $n - 1$.

$\mathcal{P} =$



$(s_1, s_2, s_3, s_4) = (1, 1, 2, 2)$



$O(P, s)$

$O(P, s)$ has h^*-polynomial

$$1 + 9z + 9z^2 + z^3$$

Conjecture (Brändén, Leander; 2016). $h^*(O(\mathcal{P}, s); z)$ is unimodal for $O(\mathcal{P}, s)$ as above.

Corollary (Gustafsson, LS; 2018). The conjecture is true if $\mathcal{P}$ has a unique minimal element.

A Second Corollary to Betke and McMullen:

For $p(z) = p_0 + p_1 z + \cdots + p_d z^d \in \mathbb{R}[z]$, there exist unique polynomials $a(z)$ and $b(z)$ of degree at most d and $d-1$, respectively, such that

- $p(z) = a(z) + zb(z)$,
- $a(z)$ is symmetric with respect to d, and
- $b(z)$ is symmetric with respect to $d-1$.

Definition. We call the ordered pair of polynomials (a, b) the *I*-**decomposition** of p.

Corollary (Betke, McMullen; 1985). If P is a lattice polytope of dimension d containing an interior lattice point, then the polynomials in the I-decomposition of $h^*(P; z)$ have only nonnegative coefficients.

If the polynomials in the I-decomposition of $p(z)$ have only nonnegative coefficients and are unimodal, then $p(z)$ is **alternatingly increasing**; i.e.

$$p_0 \leq p_d \leq p_1 \leq p_{d-1} \leq \cdots \leq p_{\lfloor \frac{d+1}{2} \rfloor}.$$

(Beck, Jochemko, and McCullough; 2016) proved:

- The h^*-polynomial of any combinatorially positive valuation of a **lattice zonotope** is real-rooted.
- The h^*-polynomial of any combinatorially positive valuation of a **centrally symmetric lattice zonotope** is alternatingly increasing.

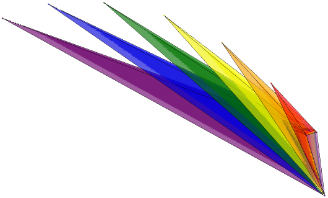
Question 1. For a generating polynomial p with I-decomposition (a, b), when are a and b both real-rooted?

One Example (LS; 2017): The base r-simplices for numeral systems have real-rooted a and b. Moreover, b **interlaces** a!

The **base-r n-simplex** is the convex hull

$$\mathcal{B}_{r,n} = \mathrm{conv}(e_1, \ldots, e_n, -q) \subset \mathbb{R}^n,$$

where $q = ((r-1)r^{i-1})_{i=1}^n \in \mathbb{R}^n$ for $r \in \mathbb{Z}_{>1}$.



Question 2. For a generating polynomial p with I-decomposition (a, b), does b interlace a?

One Answer to Question 1:

Theorem (Brändén, LS; 2018). Let

$$i(t) = \sum_{i=0}^{d} c_i t^i (t+1)^{d-i},$$

where $c_i \geq 0$ for $0 \leq i \leq d$, and

$$\sum_{n \geq 0} i(t) z^n = \frac{h(z)}{(1-z)^{d+1}}.$$

If

$$c_0 + c_1 + \cdots + c_i \leq c_d + c_{d-1} + \cdots + c_{d-i}$$

for all $0 \leq i \leq d/2$, then the I-decomposition of $h(z)$ w.r.t. d is real-rooted.

Applications:

- The h^*-polynomial for any combinatorially positive valuation of a **lattice zonotope with interior lattice points** have real-rooted I-decompositions.
- The I-decomposition of the h-polynomial of a **barycentric subdivision** of a **Cohen-Macaulay simplicial complex** is real-rooted.
- Moreover, each of these polynomials is alternatingly increasing.

An Answer to Question 2:

Theorem (Brändén, LS; 2018). Let $p \in \mathbb{R}[x]$ be degree d with I–decomposition (a, b) w.r.t. d, such that a and b have only nonnegative coefficients. If a and b have degree d and $d-1$, respectively, then the following are equivalent:

1. b interlaces a,
2. a interlaces p,
3. b interlaces p,
4. The reciprocal of p interlaces p.

Applications:

- The h^*-polynomial for any combinatorially positive valuation of a centrally symmetric lattice zonotope has I-decomposition (a, b) with b interlacing a.
- The **colored Eulerian polynomials** and **colored derangement polynomials** all have I-decomposition (a, b) with b interlacing a.
- The latter answers a conjecture **(Athanasiadis; 2017)** on the real-rootedness of the local h-polynomial of the r^{th} edgewise subdivision of the barycentric subdivision of a simplex.

Thank You for Listening!

Some References:

- P. Brändén and L. Solus. *Symmetric Decompositions and Real-rootedness*. Preprint available soon!
- N. Gustafsson and L. Solus. *Derangements, Ehrhart Theory, and local h-polynomials*. arXiv number: 1807.05246.
- L. Solus. *Simplices for Numeral Systems*. To appear in Transactions of the American Mathematical Society (2017).