On k-normality and Regularity of Normal Projective Toric Varieties

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k-normality

Definition

Let X be an irreducible projective variety and L a very ample line bundle on X, defining an embedding $X \to \mathbb{P}^r = \mathbb{P}(\mathrm{H}^0(X, L))$. We say that (the embedding of) X is k-normal if the restriction map

$$\mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to \mathrm{H}^0(X, \mathcal{O}_X(k))$$

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The k-normality of X, denoted by k_X, is defined as

$$k_X = \min\{n \in \mathbb{N} | X \text{ is } k\text{-normal for all } k \geq n\}.$$

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Now let X be a toric variety. Then L corresponds to a lattice polytope $P := P_L \subset M_{\mathbb{R}}$.

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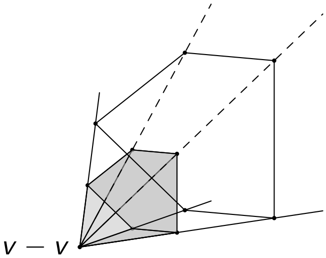
A polytope $P \subset M_{\mathbb{R}}$ is very ample if $(P - v) \cap M$ generates $\mathbb{R}_{\geq 0}(P - v) \cap M$.



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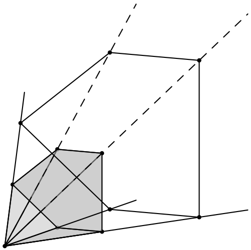

Fact: L is very ample if and only if P is very ample.

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- X is k-normal iff P is k-normal; i.e., $k_X = k_P$.
- P is very ample iff P is k-normal for k big enough (but how big?).
- (Hendelman, 1990) P is k-normal does not implies P is $(k+1)$-normal in general. Question: Is it true if P is very ample?
- k_P is not bounded by $\dim P$.

Motivation

For any projective variety $X \subset \mathbb{P}^r = \mathbb{P}(\mathrm{H}^0(X, L))$, X is k-regular (i.e., $\mathrm{H}^i(X, \mathcal{I}_X(k-i)) = 0$ for all $i > 0$) if and only if X is $(k+1)$-normal and $\mathcal{O}_X$ is k-regular. In other words,

$$\mathrm{reg}(X) = \max\{\mathrm{reg}(\mathcal{O}_X), k_X\} + 1.$$

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Lemma (Hering, 2006)

Let (X, L) be a projective polarized toric variety with L very ample. $P := P_L$. Then

$$\mathrm{reg}(\mathcal{O}_X) = \deg(P),$$

where $\deg(P)$ *is the degree of the Ehrhart h^*-polynomial of P.*

Motivation (cont'd)

As a consequence:

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We have a straightforward application of the above identity:

Proposition (T.,2018)

Let (X, L) be a polarized toric variety such that L is very ample and $(Y, L|_Y)$ a T-invariant subvariety of X. Then

$$\mathrm{reg}(X) \geq \mathrm{reg}(Y).$$

Motivation (cont'd)

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It is wrong in general, with some counterexamples recently given by McCullough & Peeva (2017). However, the conjecture is still open for toric varieties.

Motivation (cont'd)

Some known bounds:

- (Mumford, 92): $X \subset \mathbb{P}^r$ a reduced smooth subscheme purely of dimension d in characteristics 0,

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$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 2 \text{ if } d = 3$$

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- (Sturmfels, 1995) X a projective toric variety in $\mathbb{P}^{r-1}$,
$$\operatorname{reg}(X) \leq r \cdot \deg(X) \cdot \operatorname{codim}(X).$$

The toric case

Let (X, L) be a polarized toric variety, L very ample, $P = P_L$, $\dim P = d$. Then

- $\deg(X) = \mathrm{Vol}(P) = d! \cdot \mathrm{vol}(P)$, the normalized volume of P.
- $\mathrm{codim}(X) = |P \cap M| - d - 1$.

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Hence, to verify the Eisenbud-Goto conjecture for toric variety, we need to check that for any very ample lattice polytope P, is it true that

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Proposition (Hofscheier, Katthän, Nill, 2017)

Let P be a spanning lattice polytope, then

$$\deg(P) \leq \mathrm{Vol}(P) - |P \cap M| + d + 1.$$

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We have a special case as follow:

Proposition (T., 2018)

Let P be a non-hollow very ample lattice simplex. Then

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As a corollary, we obtain:

Corollary (T.,2018)

The Eisenbud-Goto conjecture holds for any polarized weighted projective space $(\mathbb{P}(q_0, \ldots, q_n), L) \subset \mathbb{P}^r$ such that L is very ample.

Main...wait for it...

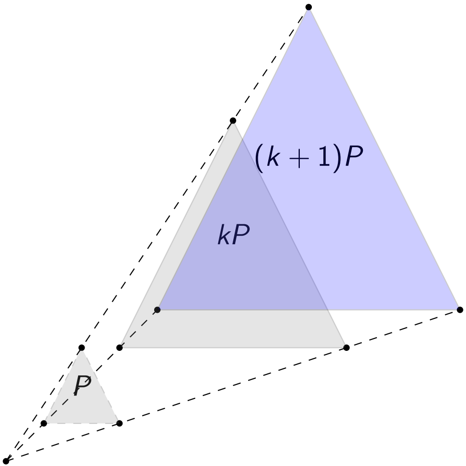
We will need some definitions for the main result. Let us start with a lemma:

Lemma

Let P be a d-dimensional lattice polytope that has n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$.

- *(Ewald-Wessels, 1991) For $k \geq d - 1$, we have*
$$(k+1)P \cap M = P \cap M + kP \cap M.$$
- *For any $k \geq n - 1$,*
$$(k+1)P \cap M = \mathcal{V} + kP \cap M.$$

...wait...



...wait...

From the previous lemma, we can define:

Definition

Let d_P be the smallest positive integer such that

$$P \cap M + kP \cap M \twoheadrightarrow (k+1)P \cap M$$

for all $k \geq d_P$. Similarly, let ν_P be the smallest positive integer such that

$$\mathcal{V} + kP \cap M \twoheadrightarrow (k+1)P \cap M$$

for all $k \geq \nu_P$.

...still wait...

Now let P be a very ample polytope. Then for any lattice point $x \in d_P \cdot P \cap M$ and vertex $d_P \cdot v$ of $d_P \cdot P$ we can define

$$\sigma(x, d_P v) = \min \left\{ n \in \mathbb{N} \middle| x - d_p v = \sum_{i=1}^{n} (w_i - v), w_i \in P \cap M \right\}.$$

and

$$m_P = \max \left\{ \sigma(x, d_P v) \mid x \in (d_P P) \cap M, \ v \text{ a vertex of } P \right\}.$$

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Lemma (T.,2017)

- $\nu_P \geq d_P$ *for any polytope* P.
- $m_P \geq d_P$ *if* P *is very ample. The equality occurs if and only if* P *is normal.*

...result, finally!

Theorem (T., 2017)

Let $P \subset M_{\mathbb{R}}$ be a very ample lattice polytope with n vertices. Then if P is not normal, then

$$k_P \leq (m_P - d_P - 1) \cdot n + \nu_P + 1.$$

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Proposition (T., 2018)

Let P be a very ample d-simplex. Then

$$k_P \leq d_P + \nu_P - 1.$$

As a consequence,

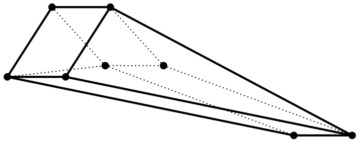
$$k_P \leq \mathrm{Vol}(P) - |P \cap M| + \frac{3d}{2}.$$

Example (Bruns & Gubeladze, 2009)

Consider the polytope P which is the convex hull of the vertices given by the columns of the following matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & s & s+1 \end{pmatrix},$$

where $s \geq 4$.



Example (cont'd)

P is very ample but not smooth, $\dim P = 3$.

- $d_P = \nu_P = 2$.
- $\mathrm{Vol}(P) = s + 3$.
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The following table compares the known-bounds for k_P.

k_P	Our bound	Sturmfels, 1995	Eisenbud-Goto
$s-1$	$8s-29$	$24s+143$	$s+2$

Our bound is sharp when $s = 4$, but is weaker than the Eisendbud-Goto bound if $s \geq 5$.

Thank you for your attention.