Cayley sums and Minkowski sums of 2-convex-normal lattice polytopes

Akiyoshi Tsuchiya (Osaka University)

Algebraic and Geometric Combinatorics on Lattice Polytopes, Osaka University

Two Questions and one Theorem

Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^d$ be lattice polytopes.

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Question (Oda)

When does the equation

$$\mathcal{P} \cap \mathbb{Z}^d + \mathcal{Q} \cap \mathbb{Z}^d = (\mathcal{P} + \mathcal{Q}) \cap \mathbb{Z}^d \tag{1}$$

hold?

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When is the Minkowski sum $\mathcal{P} + \mathcal{Q}$ IDP?

Theorem

If the Cayley sum $\mathcal{P} * \mathcal{Q}$ ($\mathrm{Cayley}(\mathcal{P}, \mathcal{Q})$) is IDP, then the equation (1) holds and the Minkowski sum $\mathcal{P} + \mathcal{Q}$ is IDP.

IDP polytope

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope.
Let $n\mathcal{P} = \{n\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$ be the nth dilated polytope of $\mathcal{P}$.

Definition

We say that $\mathcal{P}$ possesses the integer decomposition property (IDP) if for any positive integer n, the following equality holds:

$$(n-1)\mathcal{P} \cap \mathbb{Z}^d + \mathcal{P} \cap \mathbb{Z}^d = n\mathcal{P} \cap \mathbb{Z}^d,$$

namely

$$n\mathcal{P} \cap \mathbb{Z}^d = \underbrace{(\mathcal{P} \cap \mathbb{Z}^d) + \cdots + (\mathcal{P} \cap \mathbb{Z}^d)}_{n},$$

Then we call $\mathcal{P}$ IDP.

Minkowski sum and Cayley sum

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be lattice polytopes.
Let $\mathbf{e}_1, \ldots, \mathbf{e}_{m-1} \in \mathbb{R}^{m-1}$ be the unit standard basis for $\mathbb{R}^{m-1}$.
Let $\mathbf{0}$ be the origin of $\mathbb{R}^{m-1}$.

Definition
The Minkowski sum of $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is

$$\mathcal{P}_1 + \cdots + \mathcal{P}_m := \{\mathbf{x}_1 + \cdots + \mathbf{x}_m : \mathbf{x}_i \in \mathcal{P}_i\}.$$

Definition
The Cayley sum of $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is

$$\mathcal{P}_1 * \cdots * \mathcal{P}_m := \text{conv}(\{\mathbf{e}_1\} \times \mathcal{P}_1, \ldots, \{\mathbf{e}_{m-1}\} \times \mathcal{P}_{m-1}, \{\mathbf{0}\} \times \mathcal{P}_m).$$

IDP for Cayley and Minkowski sum

Theorem (T)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ *be lattice polytopes.*
If $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ *is IDP, then for any subset* $(i_1, \ldots, i_k) \subset [m] := \{1, \ldots, m\}$, $\mathcal{P}_{i_1} + \cdots + \mathcal{P}_{i_k}$ *is IDP.*

Question

When is $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ *IDP?*

IDP for dilated polytope and Minkowski sum

Theorem (Bruns-Gubeladze-Trung, 1997)

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope.
Then for any positive integer $n \geq \dim(\mathcal{P}) - 1$, $n\mathcal{P}$ is IDP.

Theorem (Higashitani, 16)

Let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^d$ be lattice polytopes.
For each i, let n_i be a positive integer with $n_i \geq \dim(\mathcal{P}_i)$.
Then $n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m$ is IDP.

Question

If for each i, $n_i \geq \dim(\mathcal{P}_i)$, then is $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ IDP?
Or does there exist positive integer $k_1, \dots, k_m$ such that if for each i, $n_i \geq k_i$, then $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ IDP?

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Or does there exist positive integer $k_1, \ldots, k_m$ such that if for each i, $n_i \geq k_i$, then $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ IDP?

Answer: No.

IDP for a tuple of lattice polytopes

Let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^d$ be lattice polytopes.

Definition

We say that the tuple $(\mathcal{P}_1, \dots, \mathcal{P}_m)$ is IDP if for any subset $(i_1, \dots, i_k) \subset [m]$, it follows that

$$(\mathcal{P}_{i_1} + \cdots + \mathcal{P}_{i_k}) \cap \mathbb{Z}^d = (\mathcal{P}_{i_1} \cap \mathbb{Z}^d) + \cdots + (\mathcal{P}_{i_k} \cap \mathbb{Z}^d).$$

Note: When $m = 2$, this notion was introduced by C. Haase and J. Hofmann.

See the first theorem again

Theorem (T)

Let $\mathcal{P}_1, \dots, \mathcal{P}_m \subset \mathbb{R}^d$ be lattice polytopes.
If $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is IDP, then for any subset $(i_1, \dots, i_k) \subset [m]$.
$\mathcal{P}_{i_1} + \cdots + \mathcal{P}_{i_k}$ is IDP.

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If $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is IDP, then for any subset $(i_1, \ldots, i_k) \subset [m]$.
$\mathcal{P}_{i_1} + \cdots + \mathcal{P}_{i_k}$ is IDP.
Moreover, $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is IDP.

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Theorem (T)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ *be lattice polytopes.*
If $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ *is IDP, then for any subset* $(i_1, \ldots, i_k) \subset [m]$.
$\mathcal{P}_{i_1} + \cdots + \mathcal{P}_{i_k}$ *is IDP.*
Moreover, $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ *is IDP.*

Question

If for each i, $n_i \geq \dim(\mathcal{P}_i)$ *and if* $(n_1\mathcal{P}_1, \ldots, n_m\mathcal{P}_m)$ *is IDP, then is* $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ *IDP?*

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If $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is IDP, then for any subset $(i_1, \ldots, i_k) \subset [m]$.
$\mathcal{P}_{i_1} + \cdots + \mathcal{P}_{i_k}$ is IDP.
Moreover, $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ is IDP.

Question

If for each i, $n_i \geq \dim(\mathcal{P}_i)$ and if $(n_1\mathcal{P}_1, \ldots, n_m\mathcal{P}_m)$ is IDP, then is $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ IDP?

Answer: Yes.

I give a more general result.

2-convex-normal polytope

Definition

A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ is called 2-convex-normal if

$$2\mathcal{P} = \mathcal{P} \cap \mathbb{Z}^d + \mathcal{P}.$$

In this case, $\mathcal{P}$ is IDP.

When is $\mathcal{P}$ 2-convex-normal?

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When is $\mathcal{P}$ 2-convex-normal?

Theorem (Gubeladze, 12)

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope and set $t = \dim \mathcal{P}$. If every edge of $\mathcal{P}$ has lattice length $\geq 2t(t+1)$, then P is 2-convex-normal.

Rewrite Higashitani's result

Proposition

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope. Then for any positive integer $n \geq \dim(\mathcal{P})$, $n\mathcal{P}$ is 2-convex-normal.

Theorem (Higashitani, 16)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be 2-convex-normal lattice polytopes. Then $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ is IDP.

IDP for Cayley sums of 2-convex-normal polytopes

Theorem (T)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be 2-convex-normal lattice polytopes. Then the Cayley sum $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is IDP if (and only if) the tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$ is IDP.

Corollary

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^N$ be lattice polytopes. For each i, let n_i be a positive integer with $n_i \geq \dim(\mathcal{P}_i)$. Then the Cayley sum $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ is IDP if (and only if) the tuple $(n_1\mathcal{P}_1, \ldots, n_m\mathcal{P}_m)$ is IDP.

Level polytope

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope.
Let $\mathrm{int}(\mathcal{P})$ denote the relative interior of $\mathcal{P}$.
Set $r = \min\{n \in \mathbb{Z} : \mathrm{int}(n\mathcal{P}) \cap \mathbb{Z}^d \neq \emptyset\}$.

Definition

We say that $\mathcal{P}$ is level of index r if for any positive integer $n > r$

$$\mathrm{int}(n\mathcal{P}) \cap \mathbb{Z}^d = \mathrm{int}(r\mathcal{P}) \cap \mathbb{Z}^d + (n-r)\mathcal{P} \cap \mathbb{Z}^d.$$

In particular, if $|\mathrm{int}(r\mathcal{P}) \cap \mathbb{Z}^d| = 1$, then we call $\mathcal{P}$ Gorenstein.

Levelness for Cayley and Minkowski sum

Theorem (T)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be lattice polytopes.
Assume that $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is level of index m.
Then $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ is level of index 1.

Question

When is $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ level of index m?

Remark

$\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is Gorenstein of index m if and only if $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ is Gorenstein of index 1 [Batyrev-Nill, 08]. *However, this is not true for the level case.*

Levelness for dilated polytope and Minkowski sum

Theorem (Bruns-Gubeladze-Trung, 1997)

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope.
Then for any positive integer $n \geq \dim(\mathcal{P}) + 1$, $n\mathcal{P}$ is level of index 1.

Theorem (Higashitani, 16)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be lattice polytopes.
For each i, let n_i be a positive integer with $n_i \geq \dim(\mathcal{P}_i) + 1$
Then $n_1\mathcal{P}_1 + \cdots + n_m\mathcal{P}_m$ is level of index 1.

Question

If for each i, $n_i \geq \dim(\mathcal{P}_i) + 1$, then is $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ level of index m?

Levelness for dilated polytope and Minkowski sum

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If for each i, $n_i \geq \dim(\mathcal{P}_i) + 1$, then is $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ level of index m?

Answer: Yes.

I give a more general result.

2-convex-level polytope

Definition

A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ with interior lattice points is called 2-convex-level if

$$\mathrm{int}(2\mathcal{P}) = \mathcal{P} \cap \mathbb{Z}^d + \mathrm{int}(\mathcal{P}).$$

In this case, $\mathcal{P}$ is level of index 1.

When is $\mathcal{P}$ 2-convex-level?

Proposition

Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice polytope. Then for any positive integer $n \geq \dim(\mathcal{P}) + 1$, $n\mathcal{P}$ is 2-convex-level.

Theorem (Higashitani)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be 2-convex-level lattice polytopes. Then $\mathcal{P}_1 + \cdots + \mathcal{P}_m$ is level of index 1.

Levelness for Cayley sums of 2-convex-level polytopes

Theorem (T)

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^d$ be 2-convex-level lattice polytopes.
Then the Cayley sum $\mathcal{P}_1 * \cdots * \mathcal{P}_m$ is level of index m.

Corollary

Let $\mathcal{P}_1, \ldots, \mathcal{P}_m \subset \mathbb{R}^N$ be lattice polytopes.
For each i, let n_i be a positive integer with $n_i \geq \dim(\mathcal{P}_i) + 1$.
Then the Cayley sum $n_1\mathcal{P}_1 * \cdots * n_m\mathcal{P}_m$ is level of index m.

THANK YOU VERY MUCH!!!