Fixed Subpolytopes of the Permutahedron

Andrés R. Vindas Meléndez

Department of Mathematics
University of Kentucky

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People

Federico Ardila
(San Francisco State Univ.)

Anna Schindler
(North Seattle College)

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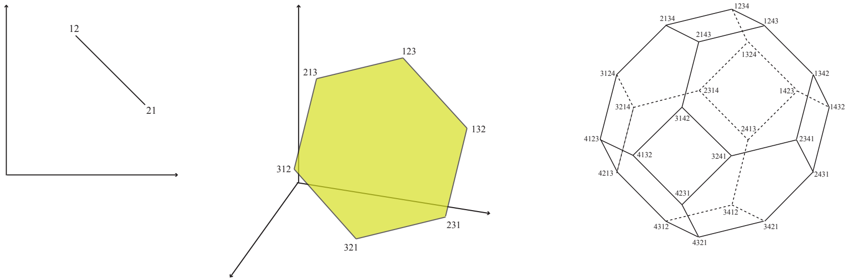
The Permutahedron

Definition

The n-permutahedron is the polytope in $\mathbb{R}^n$ whose vertices are the $n!$ permutations of $[n] := \{1, \ldots, n\}$:

$$\Pi_n := \operatorname{conv}\left\{(\pi(1), \pi(2), \ldots, \pi(n)) : \pi \in \mathfrak{S}_n\right\}.$$



The Permutahedron

The permutahedron Π_n can be described in the following three ways:

1. (Inequalities) It is the set of points $x \in \mathbb{R}^n$ satisfying
   a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, and
   b) for any proper subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1 + 2 + \cdots + k.$$

2. (Vertices) It is the convex hull of the points $(\pi(1), \ldots, \pi(n))$ as π ranges over the permutations of $[n]$.
3. (Minkowski sum) It is the Minkowski sum

$$\sum_{1 \leq j < k \leq n} [e_k, e_j] + \sum_{1 \leq k \leq n} e_k.$$

The n-permutahedron is $(n-1)$-dimensional and every permutation of $[n]$ is indeed a vertex.

Notation

- We identify each permutation $\pi \in \mathfrak{S}_n$ with the point $(\pi(1), \ldots, \pi(n))$ in $\mathbb{R}^n$. When we write permutations in cycle notation, we do not use commas to separate the entries of each cycle.
- For example, the permutation 246513 in $\mathfrak{S}_6$ is identified with the point $(2, 4, 6, 5, 1, 3) \in \mathbb{R}^6$, and write it as (1245)(36) in cycle notation.
- We assume that σ has m cycles of lengths $l_1 \geq \cdots \geq l_m$. We may assume without losing generality that $\sigma =$

$$(1 \;\; 2 \;\; \cdots \;\; l_1)(l_1{+}1 \;\; l_1{+}2 \;\; \cdots \;\; l_1{+}l_2)\cdots(l_1{+}\cdots{+}l_{m-1}{+}1 \;\; \cdots \;\; n{-}1 \;\; n).$$

- We let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and $e_S := e_{s_1} + \cdots + e_{s_k}$ for $S = \{s_1, \ldots, s_k\} \subseteq [n]$.
- We define the *cycle type* of a permutation σ to be the partition of n consisting of the lengths $l_1 \geq \cdots \geq l_m$ of the cycles of σ.

We consider Π_n under an action of the symmetric group $\mathfrak{S}_n$, where $x = (x_1, x_2, \ldots, x_n) \in \Pi_n, \quad \sigma \cdot x = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)})$.

Definition

The *subpolytope of the permutahedron* Π_n *fixed by a permutation* σ *of* $[n]$ is

$$\Pi_n^\sigma = \{x \in \Pi_n \,:\, \sigma \cdot x = x\}.$$

Fixed Subpolytopes of the Permutahedron

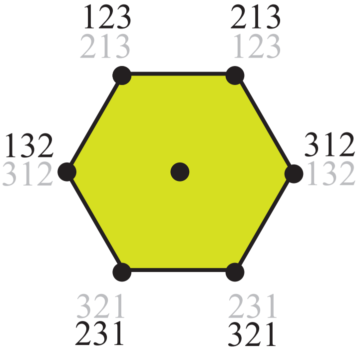
Theorem (Stapledon 2011)

Let $\mathcal{P}^g$ denote the set of lattice points of $\mathcal{P}$ that are fixed by g, i.e., $\mathcal{P}^g = \{x \in \mathcal{P} : g \cdot x = x\}$. Then

$$\mathcal{P}^g = \text{conv}\left\{ \frac{1}{|g|} \sum_{i=1}^{|g|} g^i \cdot v : v \text{ is a vertex of } \mathcal{P} \right\}$$

is a rational polytope.

Let's look at the subpolytope of Π_3 fixed by (12).



Example

1. (12) induces a reflection.

Fixed Subpolytopes of the Permutahedron

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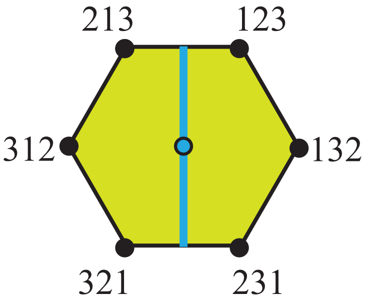
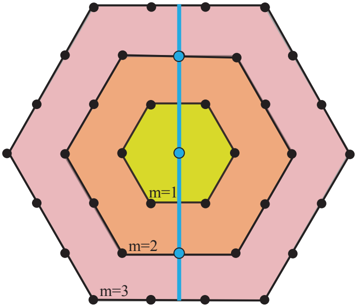


Example

- (12) induces a reflection.
- $\Pi_3{}^{(12)}$ satisfies $x_1 = x_2$.
- $\Pi_3{}^{(12)}$ is a 1-dimensional rational polytope.

Fixed Subpolytopes of the Permutahedron

Let's look at the subpolytope of Π_3 fixed by (12).



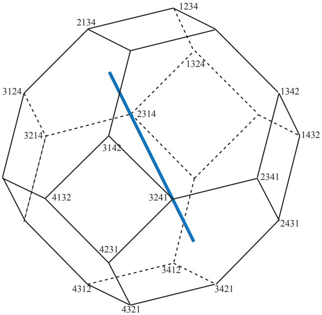
Example

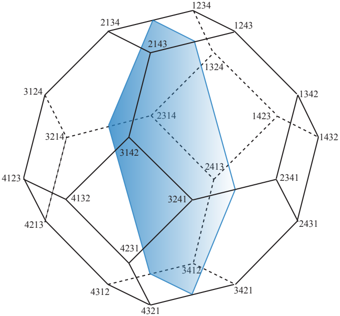
- (12) induces a reflection.
- $\Pi_3^{(12)}$ satisfies $x_1 = x_2$
- $\Pi_3^{(12)}$ is a rational polytope
- $\text{ehr}_{\Pi_3^{(12)}}(t) =$

$$\begin{cases} t+1 & \text{if } t \text{ is even,} \\ t & \text{if } t \text{ is odd} \end{cases}$$

Fixed Subpolytopes of the Permutahedron

Examples of fixed subpolytopes of Π_4





Every fixed subpolytope of Π_4 is either a point, a line segment, or a hexagon.

The Inequality Description

Proposition

For a permutation $\sigma \in \mathfrak{S}_n$, the fixed subpolytope Π_n^σ consists of the points $x \in \Pi_n$ satisfying $x_j = x_k$ for any j and k in the same cycle of σ.

Corollary

If a permutation σ of $[n]$ has m cycles then Π_n^σ has dimension $m - 1$.

Towards a Vertex Description

For a point $w \in \mathbb{R}^n$, let $\overline{w}$ be the average of the σ-orbit of w, that is,

$$\overline{w} := \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot w,$$

where $|\sigma|$ is the order of σ as an element of the symmetric group $\mathfrak{S}_n$.

Definition

Given $\sigma \in \mathfrak{S}_n$, we say a permutation $v = (v_1, \ldots, v_n)$ of $[n]$ is *σ-standard* if it satisfies the following property: for each cycle $(j_1\ j_2\ \cdots\ j_r)$ of σ, $(v_{j_1}, v_{j_2}, \ldots, v_{j_r})$ is a sequence of consecutive integers in increasing order. We define the set of *σ-vertices* to be

$$\mathrm{Vert}(\sigma) := \{\overline{w} \ : \ w \text{ is a } \sigma\text{-standard permutation of } [n]\}.$$

Towards a Vertex Description

Lemma

For any $w \in \mathbb{R}^n$, the average of the σ-orbit of w is

$$\overline{w} = \sum_{k=1}^{m} \frac{\sum_{j \in \sigma_k} w_j}{l_k} e_{\sigma_k}.$$

Corollary

The set $\mathrm{Vert}(\sigma)$ of σ-vertices consists of the $m!$ points

$$\overline{v_{\prec}} := \sum_{k=1}^{m} \left(\frac{l_k + 1}{2} + \sum_{j : \sigma_j \prec \sigma_k} l_j \right) e_{\sigma_k}$$

as $\prec$ ranges over the $m!$ possible linear orderings of $\sigma_1, \sigma_2, \ldots, \sigma_m$.

Towards a Vertex Description

For $\sigma = (1234)(567)(89)$, the σ-standard permutations in $\mathfrak{S}_9$ are

$$\begin{array}{ll}
(1,2,3,4,\ 5,6,7,\ 8,9), & (1,2,3,4,\ 7,8,9,\ 5,6), \\
(4,5,6,7,\ 1,2,3,\ 8,9), & (3,4,5,6,\ 7,8,9,\ 1,2), \\
(6,7,8,9,\ 1,2,3,\ 4,5), & (6,7,8,9,\ 3,4,5,\ 1,2),
\end{array}$$

and the corresponding σ-vertices are

$$\begin{array}{ll}
\frac{1+2+3+4}{4} e_{1234} + \frac{5+6+7}{3} e_{567} + \frac{8+9}{2} e_{89}, & \frac{1+2+3+4}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{5+6}{2} e_{8} \\
\frac{4+5+6+7}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{8+9}{2} e_{89}, & \frac{3+4+5+6}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{1+2}{2} e_{8} \\
\frac{6+7+8+9}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{4+5}{2} e_{89}, & \frac{6+7+8+9}{4} e_{1234} + \frac{3+4+5}{3} e_{567} + \frac{1+2}{2} e_{8}
\end{array}$$

Towards a Zonotope Description

Definition

Let M_σ denote the Minkowski sum

$$M_\sigma := \sum_{1\le j<k\le m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k+1}{2} e_{\sigma_k}$$

$$= \sum_{1\le j<k\le m} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^{m} \left(\frac{l_k+1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}.$$

Proposition

The zonotope M_σ is combinatorially equivalent to the standard permutahedron Π_m, where m is the number of cycles of σ.

The Descriptions of the Fixed Subpolytope are Equivalent

Theorem

The fixed subpolytope Π_n^σ can be described in the following four ways:

1. *It is the set of points x in the permutahedron Π_n such that $\sigma \cdot x = x$.*
2. *It is the set of points $x \in \mathbb{R}^n$ satisfying*
    1. $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$,
    2. *for any proper subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,*

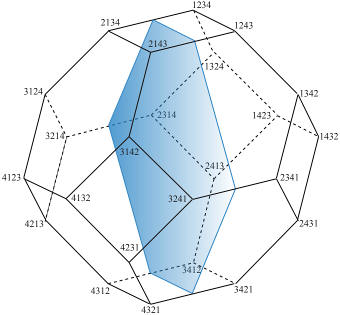
    $$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1 + 2 + \cdots + k, \ \textit{and}$$

    3. *for any i and j which are in the same cycle of σ, $x_i = x_j$.*
3. *It is the convex hull of the set $\mathrm{Vert}(\sigma)$ of σ-vertices.*
4. *It is the Minkowski sum M_σ.*

Consequently, the fixed polytope Π_n^σ is a zonotope that is combinatorially isomorphic to the permutahedron Π_m. It is $(m-1)$-dimensional and every σ-vertex is indeed a vertex of Π_n^σ.

Fixed Subpolytopes of the Permutahedron

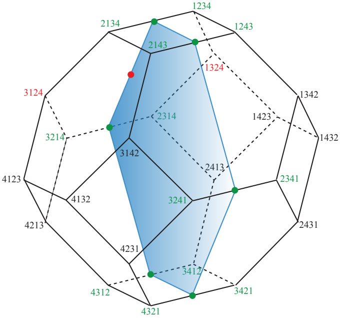
This theorem provides a vertex description for Π_n^σ, a refinement of Stapledon's description



Example

- Stapledon: $\Pi_4^{(12)} =$ $\mathrm{conv}\left\{ \frac{1}{2} \sum_{i=1}^{2} \sigma^i v : v \text{ is a vertex of } \Pi_4 \right\}$

This theorem provides a vertex description for Π_n^σ, a refinement of Stapledon's description



Example

- Stapledon: $\Pi_4^{(12)} =$ $\text{conv}\left\{ \frac{1}{2} \sum_{i=1}^{2} \sigma^i v : v \text{ is a vertex of } \Pi_4 \right\}$
- But not all of these points are vertices of $\Pi_4^{(12)}$.
- It is enough to consider the vertices with consecutive, increasing integers in positions 1 and 2.
- The number of such vertices is the number of orderings of $(12), (3)$, and (4): $3! = 6$.

The volumes of the fixed subpolytopes of Π_n

Theorem

If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the subpolytope of Π_n fixed by σ is

$$\mathrm{Vol}\,\Pi_n^\sigma = n^{m-2}\,\mathrm{gcd}(l_1, \ldots, l_m).$$

When $\sigma = \mathrm{id}$ is the identity permutation, the fixed polytope is $\Pi_n^{\mathrm{id}} = \Pi_n$, and we recover Stanley's result that $\mathrm{Vol}\,\Pi_n = n^{n-2}$.

Subpolytopes of Π_n Fixed by a subgroup of $\mathfrak{S}_n$

One might ask, more generally, for the subpolytope of Π_n fixed by a subgroup of H in $\mathfrak{S}_n$; that is,

$$\Pi_n^H = \{x \in \Pi_n \, : \, \sigma \cdot x = x \text{ for all } \sigma \in H\}.$$

It turns out that this more general definition leads to the same family of subpolytopes of Π_n.

Lemma

For every subgroup H of $\mathfrak{S}_n$ there is a permutation σ of $\mathfrak{S}_n$ such that $\Pi_n^H = \Pi_n^\sigma$.

Lattice Point Enumeration

Some subtleties already arise in the simple case when Π_n^σ is a segment; that is, when σ has only two cycles of lengths l_1 and l_2. For even t, we simply have

$$\mathrm{ehr}_{\Pi_n^\sigma}(t) = \gcd(l_1, l_2)t + 1.$$

However, for odd t we have

$$\mathrm{ehr}_{\Pi_n^\sigma}(t) = \begin{cases} \gcd(l_1, l_2)t + 1 & \text{if } l_1 \text{ and } l_2 \text{ both odd,} \\ \gcd(l_1, l_2)t & \text{if } l_1 \text{ and } l_2 \text{ have different parity,} \\ \gcd(l_1, l_2)t & \text{if } l_1 \text{ and } l_2 \text{ both even \& same 2-valuation,} \\ 0 & \text{if } l_1 \text{ and } l_2 \text{ both even \& different 2-valuation} \end{cases}$$

where the 2-*valuation* of a positive integer is the highest power of 2 dividing it.
In higher dimensions, additional obstacles arise.

The End

Gracias