A lower bound theorem for centrally symmetric simplicial polytopes

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Outline

- Basics on simplicial complexes and known theorems.
- The rigidity theory of frameworks.
- Main theorem.
- Open problems.

Simplicial complexes and polytopes

Definition

A *simplicial complex* Δ on vertex set V is a collection of subsets $\tau \subseteq V$, called faces, that is closed under inclusion.

For a simplicial complex Δ, define:

1. $\dim \tau := |\tau| - 1$ for $\tau \in \Delta$,;
2. $\dim \Delta := \max\{\dim \tau : \tau \in \Delta\}$;
3. a *facet* τ is a maximal face under inclusion;
4. the *star* of a face τ is $\mathrm{st}_\Delta \tau := \{\sigma \in \Delta : \sigma \cup \tau \in \Delta\}$;
5. the *link* of a face τ is $\mathrm{lk}_\Delta \tau := \{\sigma - \tau \in \Delta : \tau \subseteq \sigma \in \Delta\}$;

Definition

A $(d-1)$-dimensional simplicial complex Δ is a simplicial $(d-1)$-sphere if its geometric realization $||\Delta||$ is homeomorphic to a sphere of dimension $d-1$.

In particular, the boundary complex of a simplicial d-polytope is a $(d-1)$-dimensional simplicial sphere.

Definition

A simplicial complex Δ is centrally symmetric or cs if it is endowed with a free involution $\alpha : V(\Delta) \to V(\Delta)$ that induces a free involution on the set of all non-empty faces of Δ.

A simplicial d-polytope is cs if $P = -P$.

Face-number related invariants

Let Δ be a $(d-1)$-dimensional simplicial complex.

Definition

The f-number $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. The vector $(f_{-1}, f_0, \cdots, f_{d-1})$ is called the f-vector.

Definition

The h-vector of Δ, $(h_0, h_1, \cdots, h_d)$, is defined by the relation $\sum_{j=0}^{d} f_{j-1}(x-1)^{d-j} = \sum_{i=0}^{d} h_i x^{d-i}$.

Definition

The g-vector of Δ is $(g_0, g_1, \cdots, g_{\lfloor \frac{d}{2} \rfloor})$ whose entries are given by

1. $g_0 = 1$;
2. $g_i = h_i - h_{i-1}$ for $1 \le i \le \lfloor \frac{d}{2} \rfloor$.

In particular, $g_2(\Delta) = f_1(\Delta) - d f_0(\Delta) + \binom{d+1}{2}$.

The Lower Bound Theorem

A polytope is *stacked* if it can be obtained from the d-simplex by repeatedly attaching (shallow) d-simplices along facets.

Theorem (Walkup, Barnette, Billera and Lee)

Let Δ be a simplicial d-polytope for $d \geq 3$. Then $g_2 \geq 0$. Furthermore, if $d \geq 3$, then equality holds if and only if Δ is a stacked polytope.

Remarks:

1. The theorem holds even in the class of normal pseudomanifolds.
2. More recent proofs are based on rigidity theory of graphs.
3. For simplicial d-polytopes, $g_r \geq 0$ by the g-theorem.

Lower bounds for cs polytopes

Theorem (Stanley, 1987)

Let P be a cs simplicial d-polytope, where $d \geq 3$. Then $g_2(P) \geq \binom{d}{2} - d$, and more generally $g_r(P) \geq \binom{d}{r} - \binom{d}{r-1}$ for all $1 \leq r \leq d/2$.

Remarks:

1. The proof requires the Hard Lefschetz properties of polytopes.
2. Stanley also proved that $h_i \geq \binom{d}{i}$ holds for all cs $(d-1)$-dimensional Cohen-Macaulay simplicial complexes.

Infinitesimal rigidity: definitions

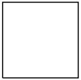
A graph $G = (V, E)$, together with a d-embedding $\mathbf{p} : V(G) \to \mathbb{R}^d$, is called a *framework* in $\mathbb{R}^d$.

Definition

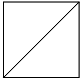
An *infinitesimal motion* of $\mathbb{R}^d$ is a map $\mathbf{\Psi} : \mathbb{R}^d \to \mathbb{R}^d$ such that for any two points $x, y \in \mathbb{R}^d$, $\frac{d}{dt}\Big|_{t=0} \left\| (x + t\mathbf{\Psi}(x)) - (y + t\mathbf{\Psi}(y)) \right\|^2 = 0$.

Definition

A framework $(G, \mathbf{p})$ is called *infinitesimally rigid* if every infinitesimal motion $\mathbf{m}$ of $(G, \mathbf{p})$ is induced by some infinitesimal motion $\mathbf{\Psi}$ of $\mathbb{R}^d$.



not inf. rigid in $\mathbb{R}^2$



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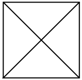

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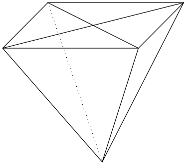
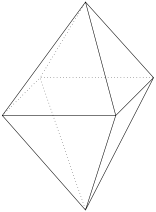


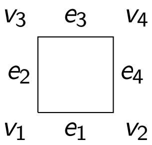
not inf. rigid in $\mathbb{R}^3$



inf. rigid in $\mathbb{R}^3$

Definition

The *rigidity matrix* $\mathrm{Rig}(G, \mathbf{p})$ of a framework $(G, \mathbf{p})$ is an $f_1(G) \times df_0(G)$ matrix with rows labeled by edges of G and columns grouped in blocks of size d, with each block labeled by a vertex of G; the row corresponding to $\{u, v\} \in E(G)$ contains the vector $\mathbf{p}(u) - \mathbf{p}(v)$ in the block of columns corresponding to u, the vector $\mathbf{p}(v) - \mathbf{p}(u)$ in columns corresponding to v, and zeros everywhere else.



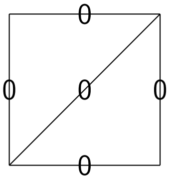
$$
\begin{array}{c} \\ \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \\ \mathbf{e_4} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \mathbf{v_1} & & \mathbf{v_2} & & \mathbf{v_3} & & \mathbf{v_4} & \end{array} \\
\begin{pmatrix}
1 & 0 & -1 & 0 & & & & \\
0 & 1 & & & 0 & -1 & & \\
 & & & & 1 & 0 & -1 & 0 \\
 & & 0 & 1 & & & 0 & -1
\end{pmatrix}
\end{array}
$$

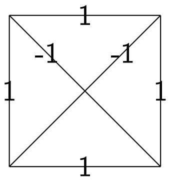
Definition

A *stress* on $(G, \mathbf{p})$ is an assignment of weights $\omega = (\omega_e \ : \ e \in E(G))$ to the edges of G such that for each vertex v,

$$\sum_{u \,:\, \{u,v\} \in E(G)} \omega_{\{u,v\}}(\mathbf{p}(v) - \mathbf{p}(u)) = \mathbf{0}.$$

We denote the space of all stresses on $(G, \mathbf{p})$ by $\mathrm{S}(G, \mathbf{p})$.





Infinitesimal rigidity: theorems

Theorem

Let $(G, \mathbf{p})$ be a framework in $\mathbb{R}^d$ that does not lie in a hyperplane of $\mathbb{R}^d$, and let $f_0(G) := |V(G)|$ and $f_1(G) := |E(G)|$. Then the following statements are equivalent:

- $(G, \mathbf{p})$ *is infinitesimally rigid in* $\mathbb{R}^d$;
- $\mathrm{rankRig}(G, \mathbf{p}) = df_0(G) - \binom{d}{2}$;
- $\dim \mathrm{S}(G, \mathbf{p}) = f_1(G) - df_0(G) + \binom{d+1}{2}$.

Properties of frameworks of polytopes:

Let $d \geq 4$ and P be a simplicial d-polytope with its natural embedding p in $\mathbb{R}^d$.

- (Whitley, 1984) $G(P)$ is infinitesimally rigid in $\mathbb{R}^d$.
- For every face τ of P with $1 \leq |V(\tau)| \leq d-3$, the framework $(\mathrm{st}_P(\tau), \mathbf{p})$ is infinitesimally rigid.
- For two vertices $u, v \in P$ such that $(\mathrm{lk}_P(u) \cap \mathrm{lk}_P(v), \mathbf{p})$ affinely spans a subspace of dimension at least $d-1$, the framework $(\mathrm{st}_P(u) \cup \mathrm{st}_P(v), \mathbf{p})$ is infinitesimally rigid.

Rigidity theory for cs graphs

From the rigidity theory of frameworks, it follows immediately that any simplicial d-polytope P, $g_2(P) \geq 0$.
What about cs simplicial d-polytopes?

Observation (Sanyal et al.)

Let $d \geq 3$ and let $(G, \mathbf{p})$ be an infinitesimally rigid cs d-framework that affinely spans $\mathbb{R}^d$. Then $g_2(G) \geq \binom{d}{2} - d$. Furthermore, if $g_2(G) = \binom{d}{2} - d$, then every stress on $(G, \mathbf{p})$ is symmetric.

Main Theorem

Theorem

Let P be a cs simplicial d-polytope with $d \geq 4$. Then $g_2(P) = \binom{d}{2} - d$ if and only if P is obtained from $\mathcal{C}_d^*$ by symmetric stacking.

Remark:

- The proof only works for simplicial d-polytopes.
- We use different proofs for the cases $d = 4$ and $d > 4$.

Proof ideas:

1. Reduce to the case when ∂P has no missing facets.
2. Check it is true for the case $d = 4$.
3. Show that for every vertex u, $\mathrm{lk}(u) \cap \mathrm{lk}(-u)$ shares $2d - 2$ vertices. Then show that $G(\mathrm{st}(u) \cup \mathrm{st}(-u)) = G(P)$.
4. Finish by face enumeration.

Open Problems

1. Does the characterization of the minimizers continue to hold in the generality of cs simplicial spheres or even cs normal pseudomanifolds?
2. Let Δ be a 2-dimensional simplicial sphere and $\{u_i, v_i\}_{i=1}^m$ is a collection of missing edges in Δ.
   Does there exists an embedding $\mathbf{p}$ of Δ such that $\mathbf{p}(u_i) = \mathbf{p}(v_i)$ for $i = 1, \ldots, m$ and $(\Delta, \mathbf{p})$ is infinitesimal rigid?
3. A generalized lower bound conjecture for cs simplicial polytopes?

Thank You!